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Concentration and Poincaré type inequalities for a degenerate pure jump Markov process

Concentración y desigualdades de tipo Poincaré para un proceso de Markov puro con salto degenerado

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Abstract We study Talagrand concentration and Poincaré type inequalities for unbounded pure jump Markov processes. In particular we focus on processes with degenerate jumps that depend on the past of the whole system, based on the model introduced by Galves and Löcherbach, in order to describe the activity of a biological neural network. As a result we obtain concentration properties.
Keywords brain neuron networks, Poincaré inequality, Talagrand inequality.

Resumen Estudiamos la concentración de Talagrand y las desigualdades de tipo Poincaré para procesos de Markov de salto puro no acotado. En particular, nos centramos en los procesos con saltos degenerados que dependen del pasado de todo el sistema, basado en el modelo introducido por Galves y Löcherbach, para describir la actividad de una red neuronal biológica. Como resultado obtenemos algunas propiedades de concentración.
Palabras Claves desigualdades de Poincaré, desigualdades de Talagrand, redes neuronales cerebrales.

1 Introduction

Our aim is to obtain Poincaré type inequalities related to the semigroup P_t and the associated invariant measure of unbounded jump processes based on a model introduced by Galves and Löcherbach (Galves & Löcherbach, 2013), in order to

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describe neural interactions. As a result, exponentially fast rates of convergence to equilibrium are obtained. There are three interesting features about this particular jump process. The first is the degenerate nature of the jumps, since every neuron jumps to zero after it spikes, and thus looses its memory. The second, is that if one focuses on one neuron, then the spiking probability depends on its current state and so from the system's past. Thirdly, the intensity function that describes the jump behaviour of any of the unbounded neurons at any time is an unbounded function.

For P_t the associated semigroup and μ the invariant measure we show the Poincaré type inequality

$$\frac{1}{c(t)}\mu\left(Var_{P_t}(f)\right) \leq \mu(\Gamma(f,f)) + \mu(F(\phi)\Gamma(f,f)\mathcal{I}_D)$$

where the second term is a local term for the compact set $D := \{x \in \mathbb{R}_+^N : x^i \leq m,\ 1 \leq i \leq N\}$, for some m. Accordingly, for every function defined outside the compact set $\{x \in \mathbb{R}_+^N : x^i \leq m+1,\ 1 \leq i \leq N\}$ we obtain the stronger

$$\mu\left(Var_{P_t}(f)\right) \leq c(t)\mu(\Gamma(f,f))).$$

Furthermore, in relation to the invariant measure the following inequality is proven

$$Var_{\mu}(f) \leq c\mu\left(\Gamma(f,f)\right).$$

Consequently, we derive concentration properties

$$\mu\left(\{P_t f - \mu(f) \geq r\}\right) \leq e^{-cr}.$$

In addition, we show Talagrand type concentration inequalities,

$$\mu\left(\left\{\sum_{i}^{N} x^i < r\right\}\right) \geq 1 - e^{-cr}.$$

In the next section the neuroscience framework is described.

1.1 The neuroscience framework

We consider a set of finitely many interacting neurons, say N in number. Each one of these neurons i, $1 \leq i \leq N$ is characterized by its membrane potential $X_t^i : \mathbb{R}_+ \to \mathbb{R}_+$ at time $t \in \mathbb{R}_+$. In this way, an N dimensional random process $X_t = (X_t^1, ..., X_t^N)$ is defined, representing the the network's membrane potential.

The neuron's membrane potential does not describe only the neuron itself, but also the interactions between the different neurons in the network, through the spiking activity of the neuron. What is called spike, or alternatively action potential, is a high-amplitude and brief depolarisation of the membrane potential that occurs from time to time, and constitutes the only disturbance of a neuron's membrane potential transmitted between the neurons.

The frequency at which a neuron spikes is expressed through the intensity function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$. $\phi(x)$ describes the intensity of a neuron with membrane potential x.

Every time a neuron spikes it loses its memory, in the sense that after a spike has occurred, the membrane potential is always reduced to zero. Then, the rest of the neurons $j \neq i$ on the system have their membrane potential increased by a positive quantity $W_{i \to j} \geq 0$ called the synaptic weight, which represents the influence of the spiking neuron i on j. It should be noted that the membrane potential of any of the N neurons remains constant between two consequent jumps.

From our discussion up to this point it should be clear that the whole dynamic of the whole interacting neural system is interpreted exclusively by the jump times. Thus, from a purely a probabilistic perspective, the dynamic can be explained by a simple point process. One should however bear in mind that since the spiking neuron jumps to zero these point processes are non-Markovian. For examples of Hawkes processes describing neural systems one can look at Chevallier (2017); Duarte, Löcherbach, & Ost (2019); Duarte & Ost (2014); Galves & Löcherbach (2013); Hansen, Reynaud-Bouret, & Rivoirard (2015); Hodara & Löcherbach (2017).

An alternative view point, instead of focusing exclusively on the jump times, is to study how the membrane potential evolves between jumps as well, when this evolution is already determined. In the case of deterministic drift between the jumps, for example, as examined in Hodara, Krell, & Löcherbach (2016), the membrane potential is attracted towards an equilibrium potential exponentially fast. In that case, the process is a Piecewise Deterministic Markov Process (PDMP). This process was introduced in Davis (1984, 1993) by Davis. PDMPs are frequentely used to model chemical and biological phenomena (see for instance André (2019); André & Planche (2021); Crudu, Debussche, Muller, & Radulescu (2012); Pakdaman, Thieullen, & Wainrib (2010), as well as Azaïs, Bardet, Génadot, Krell, & Zitt (2014) for an overview).

In the current paper we adopt a similar framework, except we do not consider drifts occurring between two consecutive jumps, but rather a pure jump Markov process, which for convenience we will abbreviate as PJMP.

Although here we work with a finite number of neurons, so that we can take advantage of the Markovian nature of the membrane potential, Hawkes processes in general allow the study of infinite neural systems, as in Galves & Löcherbach (2013) or Hodara & Löcherbach (2017).

Unlike Hodara et al. (2016), a Lyapunov-type inequality allows us to get rid of the compact state-space assumption. Due to the deterministic and degenerate nature of the jumps, the process does not have a density continuous with respect to the Lebesgue measure. We refer the reader to Löcherbach (2018) for a study of the density of the invariant measure. Here, we make use of the lack of drift between the jumps to work with discrete probabilities instead of densities.

1.2 The model

The model is similar to the compact model presented in Hodara & Papageorgiou (2019); Papageorgiou (2020), only that now we consider the case of non compact membrane potential. Consider an increasing intensity function $\phi : \mathbb{R}_+ \mapsto \mathbb{R}_+$, satisfying the conditions:

$$\phi(x) \geq \delta \tag{1}$$

and

$$\phi(x) > cx \text{ for } x \in \mathbb{R}_+. \tag{2}$$

For positive constants δ and c. The intensity function characterizes the Markov process $X_t = (X_t^1, \ldots, X_t^N)$. If we define

$$(\varDelta_i(x))^j = \begin{Bmatrix} x^j + W_{i\to j} & j \neq i \\ 0 & j = i \end{Bmatrix}, \tag{3}$$

then the process X has generator $\mathcal{L}$ which is expressed through the intensity function:

$$\mathcal{L}f(x) = \sum_{i=1}^{N} \phi(x^i)\left[f(\varDelta_i(x)) - f(x)\right] \tag{4}$$

for every $x \in \mathbb{R}_+^N$ and $f : \mathbb{R}_+^N \to \mathbb{R}$ any test function.

1.3 Poincaré type inequalities

We start with a description of the analytical framework and the definition of the Poincaré inequality on a general discrete setting. For more details one can consult Ane & Ledoux (2000), Chafaï (2004), Diaconis & Saloff-Coste (1996), Saloff (1996) and Wang & Yuan (2010). For a function f and a probability measure ν, $\int f d\nu$ will be used for the expectation of the function with respect to that measure.

Consider a Markov process $(X_t)_{t\geq 0}$ with Markov semigroup $P_t f(x) = e^x(f(X_t))$. We define

$$\mathcal{L}f := \lim_{t\to 0+} \frac{P_t f - f}{t}$$

the infinitesimal generator of the process. We will frequently use the following relationships: $\frac{d}{dt}P_t = \mathcal{L}P_t = P_t\mathcal{L}$ (see for instance Guionnet & Zegarlinksi (2003)).

Furthermore, considering a semigroup $(P_s)_{s\geq 0}$ and a measure μ, the measure is invariant to the semigroup if

$$\mu P_s = \mu, \quad \text{for every} \quad s \geq 0.$$

From the definition of the generator we obtain that $\mu(\mathcal{L}f) = 0$.

We will also consider the operator $\Gamma(\cdot,\cdot)$, usually called the "carré du champ", defined as follows

$$\Gamma(f,g) := \frac{1}{2}(\mathcal{L}(fg) - f\mathcal{L}g - g\mathcal{L}f).$$

When we consider PJMPs the operator (4) takes the form

$$\Gamma(f,f) = \frac{1}{2}\left(\sum_{i=1}^{N} \phi(x^i)\left[f(\varDelta_i(x)) - f(x)\right]^2)\right).$$

For a function f and a measure m, the variance of the function with respect to this measure is $Var_m(f) = m(f - m(f))^2$.
We say that a measure m, satisfies a Poincaré inequality if the following holds

$$Var_m(f) \leq Cm(\Gamma(f,f))$$

for a constant $C > 0$ independent of f. In the case where instead of a single measure we have a family of measures as happens with semigroups, then the constant of the inequality may depend on the time, as in the examples studied in Wang & Yuan (2010), Ane & Ledoux (2000) and Chafaï (2004). The aforementioned papers used the so-called semigroup method. Following the same approach in the current work leads to an inequality for the semigroup P_t which involves a time constant $C(t)$.

In both Wang & Yuan (2010) and Ane & Ledoux (2000), the translation property

$$e^{z+y}f(X_t) = e^z f(X_t + y)$$

was used to retrieve the carré du champ. Taking advantage of this, for example in Wang & Yuan (2010), the inequality was obtained for a constant $C(t) = t$ for a special example of point processes.

In a recent paper (Hodara & Papageorgiou, 2019) the same degenerate PJMP as in (1)-(4) was considered but for bounded neurons. The Poincaré type inequality obtained for the compact case was

$$Var_{P_t}(f(x)) \leq \alpha(t)P_t\Gamma(f,f)(x) + \beta\int_0^t P_s\Gamma(f,f)(x)ds$$

where $\alpha(t)$ is a polynomial depending on time t of order two and β some positive constant. For the compact case, this was later improved in Papageorgiou (2020) where the stronger modified log-Sobolev inequality was obtained. In the more general non compact case examined in the current paper we will prove the alternative weighted Poincaré type inequality.

$$\int Var_{P_t}(f)(x)\mu(dx) \leq \delta_1(t)\mu(\Gamma(f,f)(x)) + \delta_2(t)\mu\left(F(\phi)P_t(\Gamma(f,f)\mathcal{I}_{x_t\in D})(x)\right).$$

Consequently, the stronger

$$\int Var_{P_t}(f)(x)\mu(dx) \leq \delta_1(t)\mu(\Gamma(f,f)(x))$$

holds for every function f with a domain outside the compact $\{x \in \mathbb{R}_+^N : x_i \leq m + \text{máx}_{i=1}^N W_{i\to j},\ 1 \leq i \leq N\}$. In order to handle the intensity functions we will use the Lyapunov method presented in Cattiaux, Guillin, Wang, & Wu (2009) and Bakry, Cattiaux, & Guillin (2008) which has the advantage of reducing the problem from the unbounded to the compact case.

The inequality for the semigroup family $\{P_t, t \geq 0\}$ which refers to the general case where neurons take values in the whole of $\mathbb{R}_+$ follows.

Theorem 1. *Assume the PJMP as described in (1)-(4). Then, for every $t \geq t_1$, for some $t_1 > 0$, the following weighted Poincaré type inequality holds*

$$\int Var_{E^x}(f(x_t))\mu(dx) \leq \delta_1(t)\mu(\Gamma(f,f)(x)) + \delta_2(t)\mu\left([F(\phi)(x)]P_t(\Gamma(f,f)I_{x_t\in D})(x)\right)$$

where

$$F(\phi)(x) = \sum_{i=1}^{N} \phi(x^i),$$

while $\delta_1(t)$ a first and $\delta_2(t)$ a third order polynomial of t respectively, that do not depend on the function f.

As a direct corollary of the theorem we obtain the following.

Corollary 1. *Assume the PJMP as described in (1)-(4). Then, for every function f with a domain outside $\{x \in \mathbb{R}_+^N : x_i \leq m + \text{máx}_{i=1}^N W_{i\to j},\ 1 \leq i \leq N\}$, the following Poincaré type inequality holds*

$$\mu\left(Var_{P_t}(f)\right) \leq c(t)\mu(\Gamma(f,f)))$$

for every $t \geq t_1$, for some $t_1 > 0$.

The invariant measure is presented on the next theorem.

Theorem 2. *Assume the PJMP as described in (1)-(4). Then μ satisfies a Poincaré inequality*

$$\mu\,(f-\mu f)^2 \leq C_0\mu(\Gamma(f,f))$$

for some constant $C_0 > 0$.

1.4 Concentration and other Talagrand type inequalities

Concentration inequalities play a vital role in the examination of a system's convergence to equilibrium. Talagrand (see Talagrand (1995) and Talagrand (1991)) associated the log-Sobolev and Poincaré inequalities (see for instance Guionnet & Zegarlinksi (2003); Bakry, Gentil, & Ledoux (2014); Inglis & Papageorgiou (2014)) for exponential distributions with concentration properties (see also (Bobkov & Ledoux, 1997)), that is

$$\mu\left(|P_t f - \mu(f)| > r\right) \leq \lambda_0 e^{\lambda\mu(F)} e^{-\lambda r^p} \tag{5}$$

for some $p \geq 1$. In particular, when the log-Sobolev inequality holds, then (5) is true for $p = 2$, while in the case of the weaker Poincaré inequality, the exponent is $p = 1$. Furthermore, the modified log-Sobolev inequality that interpolates between the two, investigated for example in Barthe & Roberto (2008); Gentil, Guillin, & Miclo (2005); Papageorgiou (2011), gives convergence to equilibrium of speed $1 < p < 2$.

Concentration properties related to the Poincaré inequality, or as in our case, the Poincaré type inequality, are closely related with exponential integrability of the measure, that is

$$\mu(e^{\lambda f}) < +\infty$$

for some appropriate class of functions f. This problem, is itself connected to bounding the carré du champ of the exponent of a function

$$\mu(\Gamma(e^{\lambda f/2}, e^{\lambda f/2})) \leq \frac{\lambda^2}{4}\Psi(f)\mu(e^{\lambda f}) \tag{6}$$

for some $\Psi(f)$ uniformly bounded. When diffusion processes are considered where the carré du champ is defined through a derivation, (6) is satisfied for $\|\nabla f\|_\infty < 1$ (more details on section 4). The subject is thoroughly discussed in Ledoux (1999). In our case we consider

$$|||f|||_\infty = \sup\left\{\mu(fg);\ g : \mu(g) \leq 1\right\}.$$

Then we can obtain exponential integrability and a bound (6) for functions f such that $|||\phi(x^i)D(f)^2|||_\infty < 1$ and $|||\phi(x^i)e^{\lambda D(f)}D(f)^2|||_\infty < 1$, where $D(f)(x) = \sup_{i=1}^N |f(x) - f(\Delta_i(x))|$. The main concentration property for this class of functions follows.

Theorem 3. *Assume the PJMP as described in (1)-(4). For every function f, such that $\mu(f) < \infty$, satisfying*

$$|||\phi(x^i)D(f)^2|||_\infty < 1 \ \ \textit{and} \ \ |||\phi(x^i)e^{\lambda D(f)}D(f)^2|||_\infty < 1$$

there exists a constant $\lambda_0 > 0$ such that

$$\mu\left(\{|f - \mu(f)| > r\}\right) \leq \lambda_0 e^{\lambda\mu(f)} e^{-\lambda r},$$

for some $\lambda, \lambda_0 > 0$*.*

Consequently we obtain the following convergence to equilibrium property:

Corollary 2. *Assume* $\mu(f) < \infty$*. Assume the PJMP as described in (1)-(4). For every function* f*, satisfying*

$$|||\phi(x^i)D(f)^2|||_\infty < 1 \quad and \quad |||\phi(x^i)e^{\lambda D(f)}D(f)^2|||_\infty < 1$$

there exist constants λ, λ_0 *such that*

$$\mu\left(\{|P_t f - \mu(f)| > r\}\right) \leq \lambda_0 e^{\lambda\mu(f)} e^{-\lambda r}$$

where μ *is again the invariant measure of the semigroup.*

Furthermore, for the case of unbounded neurons, we can obtain Talagrand inequalities similar to the ones proven for the modified log-Sobolev inequality in Barthe & Roberto (2008).

Theorem 4. *Assume the PJMP as described in (1)-(4). Then the following Talagrand inequality holds.*

$$\mu\left(\left\{x : \sum_i x^i \leq r\right\}\right) \geq 1 - \lambda_0 e^{-\lambda r}$$

for some $\lambda_0, \lambda > 0$*.*

2 Methodology

The methodology is developed around the need to control the variance outside a compact domain. In order to prove the Poincaré inequality for the non-compact case there are two main steps. At first we restrict our functions on some compact domain and we prove a local weighted Poincaré inequality. In order to expand the result to the non-compact case we then show a Lyapunov inequality, which places a barrier on the values of the variance when the functions are defined beyond a compact set. Using the two, we obtain a weighted Poincaré inequality for the non-compact case. The second step is to prove the actual Poincaré inequality for the invariant measure.

Having obtained the Poincaré inequality for the invariant measure we prove the Talagrand concentration properties.

The paper is structured in the following way. At first the proof of the Poincaré inequality for the semigroup P_t and the invariant measure μ are presented in sections 3.1 and 3.2 respectively. For both inequalities a Lyapunov inequality will be used to control the behaviour of the neurons outside a compact set. This is proven at the beginning of section 3. In the final section 4 the concentration inequalities are proven.

3 Proof of the Poincaré inequalities

In both the inequalities involving the semigroup and the invariant measure, the use of a Lyapunov function will be a crucial tool in order to control the intensity function outside a compact set.

We recall that under the framework of Hodara et al. (2016), our generator is defined by

$$\mathcal{L}f(x) = \sum_{i=1}^{N} \phi(x^i)\left(f\left(\Delta_i(x)\right) - f(x)\right)$$

where $\Delta_i(x)$ is defined by $(\Delta_i(x))^j := x_j + W_{i\to j}$ if $j \neq i$ and $(\Delta_i(x))^i := 0$.

Since we assume that for all neurons i and j, $W_{i\to j} \geq 0$, we can then consider that the state space is $\mathbb{R}_+^N$.

We put $W_i := \sum_{j\neq i} W_{i\to j}$.

Lemma 1. *Assume that for all $x \in \mathbb{R}_+$, $\phi(x) \geq cx$ and $\delta \leq \phi(x)$ for some constants c and $\delta > 0$. Then if we consider the Lyapunov function:*

$$V(x) = 1 + \sum_{i=1}^{N} x^i$$

there exist positive constants ϑ, b and m so that the following Lyapunov inequality holds

$$\mathcal{L}V \leq -\vartheta V + b\mathcal{I}_B$$

for the set $B = \left\{\sum_{i=1}^{N} x^i \leq m\right\}$.

Proof. For the Lyapunov function V as stated before, we have

$$\begin{aligned}
\mathcal{L}V(x) &= \sum_{i=1}^{N} \phi(x^i)(W_i - x^i) \\
&= \sum_{i:x^i>1+W_i} \phi(x^i)(W_i - x^i) + \sum_{i:x^i\leq 1+W_i} \phi(x^i)(W_i - x^i) \\
&\leq -\sum_{i:x^i>1+W_i} \phi(x^i) + \sum_{i:x^i\leq 1+W_i} \phi(1+W_i)W_i - \delta \sum_{i:x^i\leq 1+W_i} x^i \\
&\leq -(c\wedge\delta)\sum_{i=1}^{N} x^i + \sum_{i=1}^{N} \phi(1+W_i)W_i.
\end{aligned}$$

Putting $b := \sum_{i=1}^{N} \phi(1+W_i)W_i$, we have for any $\alpha \in [0,1]$

$$\begin{aligned}\mathcal{L}V(x) &\le -\alpha(c\wedge\delta)\sum_{i=1}^{N}x^i-(1-\alpha)(c\wedge\delta)\sum_{i=1}^{N}x^i+b\\ &\le -\alpha(c\wedge\delta)V(x)+b+\alpha(c\wedge\delta)-(1-\alpha)(c\wedge\delta)\sum_{i=1}^{N}x^i\\ &\le -\alpha(c\wedge\delta)V(x)+(b+\alpha(c\wedge\delta))\,1_B(x),\end{aligned}$$

with

$$B=\left\{\sum_{i=1}^{N}x^i\le\frac{b+\alpha(c\wedge\delta)}{(1-\alpha)(c\wedge\delta)}\right\}$$

in which case $m=\frac{b+\alpha(c\wedge\delta)}{(1-\alpha)(c\wedge\delta)}$.

3.1 Poincaré inequality for the semigroup

In Theorem 1 presented in the current section we prove the main results of the paper for systems of neurons that take values on $\mathbb{R}_+$.

As already explained, the approach used will be to reduce the problem from the unbounded case to the compact case examined in Hodara & Papageorgiou (2019). To do this we will follow closely the Lyapunov approach developed in Cattiaux et al. (2009); Bakry et al. (2008) to prove super Poincaré inequalities.

We start by showing that the chain returns to the compact set D with a strictly positive probability bounded from below.

For a neuron $i\in I$ and time s, we define $p_s(x)$ to be the probability that the process starting with initial configuration x has no jump before time s, and $p^i_s(x)$ the probability that during time s only the neuron i jumps. Then, (see also Hodara & Papageorgiou (2019))

$$p_s(x)=e^{-s\overline{\phi}(x)}$$

and

$$\begin{aligned}p^i_s(x)&=\int_0^s\phi(x^i)e^{-u\overline{\phi}(x)}e^{-(s-u)\overline{\phi}(\Delta_i(x))}du\\ &=\left\{\begin{array}{ll}\frac{\phi(x^i)}{\overline{\phi}(x)-\overline{\phi}(\Delta_i(x))}\left(e^{-s\overline{\phi}(\Delta_i(x))}-e^{-s\overline{\phi}(x)}\right) & \text{if } \overline{\phi}(\Delta_i(x))\neq\overline{\phi}(x)\\ s\phi(x^i)e^{-s\overline{\phi}(x)} & \text{if } \overline{\phi}(\Delta_i(x))=\overline{\phi}(x)\end{array}\right\}\end{aligned}\tag{7}$$

where above we have denoted $\overline{\phi}(x)=\sum_{j\in I}\phi(x_j)$. Furthermore, if we denote

$$t_0=\left\{\begin{array}{ll}\frac{ln\left(\overline{\phi}(x)\right)-ln\left(\overline{\phi}(\Delta_i(x))\right)}{\overline{\phi}(x)-\overline{\phi}(\Delta_i(x))} & \text{if } \overline{\phi}(\Delta_i(x))\neq\overline{\phi}(x)\\ \frac{1}{\overline{\phi}(x)} & \text{if } \overline{\phi}(\Delta_i(x))=\overline{\phi}(x)\end{array}\right\}\tag{8}$$

then $p_s^i(x)$ as a function of the time s, is continuous, strictly increasing on $(0, t_0)$ and strictly decreasing on $(t_0, +\infty)$, while we have $p_0^i(x) = 0$.

For any configuration $y \in D$ we define the set of configurations D_y containing all configurations x such that for some $t > 0$, $\pi_t(x, y) := P_x(X_t = y) > 0$.

Lemma 2. *Assume the PJMP as described in (1)-(4). Then, for every $y \in D$ in the domain of the invariant measure and $x \in D_y$, there exist a $\theta > 0$ and a $t_1 > 0$, such that*

$$\pi_t(x, y) \geq \frac{1}{\theta}$$

for every $t \geq t_1$.

Proof. We want to show that for every configuration $y \in D$ in the domain of the invariant measure, one has that $\pi_t(x, y) \geq \frac{1}{\theta}$ for some positive θ. The proof will be divided in three parts.

A) At first, for $y \in D$, we restrict ourselves to every $x \in D \cap D_y$.

Since $\mu(y) > 0$ and $\lim_{t\to\infty} \pi_t(x, y) = \mu(y)$ we readily obtain that for every couple $x, y \in D$ there exist $\theta_1 > 0$ and $t_{x,y} > 0$ such that for every $t > t_{x,y}$ we have that $\pi_t(x, y) > \frac{1}{\theta_1}$. But since D is compact, the configurations in D are finite in number and so $\max_{x,y\in D}\{t_{x,y}\} < \infty$. We thus conclude that there exists a $\theta_1 > 0$ such that

$$\pi_t(x, y) > \frac{1}{\theta_1},$$

for every $t > t_1' := max_{x,y\in D}\{t_{x,y}\}$.

In the next two steps we extend the last result to $x \in D^c$.

B) We will show that there exist $\theta_2 > 0$ and $\delta^{-1} \geq t_2 > 0$, such that for every $x \in D^c \cap D_y$, there exists a $z \in D \cap D_y$ such that

$$\pi_{t_2}(x, z) \geq \frac{1}{\theta_2}.$$

We enumerate the N neurons with numbers from 1 to N in decreasing order, so that $\phi(x_i) \geq \phi(x_{i+1})$. Define $\hat{x}^i = \Delta_i(\Delta_{i-1}(...\Delta_1(x))...)$ the configuration starting from x after the $1st$, then the $2nd$ up to the time the i'th neuron has spiked in that order. Then for every $s_i > 0$ we have

$$\pi_{t_2}(x, \hat{x}^N) \geq p_{s_1}^1(x) p_{s_2}^2(\hat{x}^1) ... p_{s_N}^N(\hat{x}^{N-1}),$$

where we recall that $p_s^i(x)$ is the probability that only the neuron i jumps during time s. If we choose $s_i = \frac{1}{N\phi(\hat{x}_i^{i-1})}$ then we have $p_{s_i}^i(\hat{x}^{i-1}) \geq N^{-1}e^{-1}$. To see this, from (7) we can compute bounds for $p_{s_i}^i(\hat{x}^{i-1})$. In the case where $\overline{\phi}(\Delta^i(\hat{x}^{i-1})) = \overline{\phi}(\hat{x}^{i-1})$ we have

$$p_{s_i}^i(\hat{x}^{i-1}) = s_i \phi(\hat{x}_i^{i-1}) e^{-s_i \overline{\phi}(\hat{x}^{i-1})} \geq N^{-1} e^{-1},$$

since $\phi(\hat{x}_i^{i-1}) \geq \phi(\hat{x}_j^{i-1})$ for every $j \neq i$, implies that $\frac{\overline{\phi}(\hat{x}^{i-1})}{N\phi(\hat{x}_i^{i-1})} \leq 1$. In the opposite case where $\overline{\phi}(\Delta^i(\hat{x}^{i-1})) \neq \overline{\phi}(\hat{x}^{i-1})$, we then have

$$p^i_{s_i}(\hat{x}^{i-1}) = \frac{\phi(\hat{x}^{i-1}_i)}{\overline{\phi}(\hat{x}^{i-1}) - \overline{\phi}(\Delta^i(\hat{x}^{i-1}))}\left(e^{-s_i\overline{\phi}(\Delta^i(\hat{x}^{i-1}))} - e^{-s_i\overline{\phi}(\hat{x}^{i-1})}\right)$$
$$\geq \frac{s_i\phi(\hat{x}^{i-1}_i)}{\overline{\phi}(\hat{x}^{i-1}) - \overline{\phi}(\Delta^i(\hat{x}^{i-1}))} e^{-s_i\,\text{máx}\{\overline{\phi}(\Delta^i(\hat{x}^{i-1})),\overline{\phi}(\hat{x}^{i-1})\}}\left(\overline{\phi}(\hat{x}^{i-1}) - \overline{\phi}(\Delta^i(\hat{x}^{i-1}))\right)$$
$$\geq \frac{1}{N}e^{-1},$$

since $\frac{\overline{\phi}(\Delta^i(\hat{x}^{i-1}))}{N\phi(\hat{x}^{i-1}_i)} \leq 1$ and $\frac{\overline{\phi}(\hat{x}^{i-1})}{N\phi(\hat{x}^{i-1}_i)} \leq 1$.

So we obtain

$$\pi_{t_2}(x, z) \geq (Ne)^{-N},$$

and the result is proven for $\theta_2 = (Ne)^N$, $z = \hat{x}^N$ and $t_2 = \sum_{i=1}^N s_i \leq \frac{1}{\delta}$.

C) Having shown (A) and (B) we can finish the proof for $x \in D^c$. For this, it is sufficient, for every $y \in D$ and $x \in D^c \cap D_y$ to write

$$\pi_t(x, y) \geq \pi_{t_2}(x, \hat{x}^N)\pi_{t-t_2}(\hat{x}^N, y)$$

and the assertion follows for $t \geq \frac{1}{\delta} + t_2$. Consequently, the Lemma follows for $t \geq t_1 := max\{t'_1, t_2 + \frac{1}{\delta}\}$.

This Lemma will be used to show a key result for the proof of the local Poincaré inequality.

Lemma 3. *Assume $z \in D^c$. For the PJMP as described in (1)-(4), we have*

$$\left(\int_0^{t-s}\left(\mathrm{e}^{\Delta_i(z)}(\mathcal{L}f(z_u)\mathcal{I}_{z_u\in D}) - \mathrm{e}^{z}(\mathcal{L}f(z_u)\mathcal{I}_{z_u\in D})\right)du\right)^2 \leq 4\theta^2 t^2 M \mathrm{e}^{z}(\Gamma(f,f)(x_t)\mathcal{I}_{x_t\in D})$$

for every $t \geq t_1$.

Proof. We can compute

$$\mathrm{i}_2 := \left(\int_0^{t-s}\left(\mathrm{e}^{\Delta_i(z)}(\mathcal{L}f(z_u)\mathcal{I}_{z_u\in D}) - \mathrm{e}^{z}(\mathcal{L}f(z_u)\mathcal{I}_{z_u\in D})\right)du\right)^2 \leq$$
$$2\left(\int_0^{t-s}\sum_{y\in D}\pi_u(\Delta_i(z), y)|\mathcal{L}f(y)|du\right)^2 + 2\left(\int_0^{t-s}\sum_{y\in D}\pi_u(z, y)|\mathcal{L}f(y)|du\right)^2$$

Since $t \geq t_1$, we can use Lemma 2 to bound $\pi_u(w, y) \leq \theta\pi_t(z, y)$ for every $y \in D$ and $\omega \in \{z, \Delta_i(z)\}$. We obtain

$$\begin{aligned}\mathrm{i}_2 \leq & 4\theta^2 \left(\int_0^{t-s} \sum_{y\in D} \pi_t(z,y) |\mathcal{L} f(y)| du \right)^2 \\ = & 4\theta^2 t^2 \left(\sum_{y\in D} \pi_t(z,y) \left(\sum_{i=1}^{N} \phi(y^i) |f(\Delta_i(y)\right) - f(y)|) \right)^2 .\end{aligned}$$

Using below two times the Cauchy-Schwarz inequality we get

$$\begin{aligned}\mathrm{i}_2 \leq & 4\theta^2 t^2 M \sum_{y\in D} \pi_t(z,y) \sum_{i=1}^{N} \phi(y^i) \left(f(\Delta_i(y)) - f(y)\right)^2 \\ = & \ 4\theta^2 t^2 M \mathrm{e}^z(\Gamma(f,f)(x_t)\mathcal{I}_{x_t\in D}),\end{aligned}$$

where $M := N^2(\phi(m) + 1)^2$.

Lemma 4. *For the PJMP as described in (1)-(4), we have*

$$\begin{aligned}\mathrm{e}^x(f^2(x_t)\mathcal{I}_{x_t\in D}) - (\mathrm{e}^x(f(x_t)\mathcal{I}_{x_t\in D}))^2 \leq & 2t\Gamma(f,f)(x) + \\ & 8\theta^2 t^3 M \left(\sum_{i=1}^{N} \phi(x^i) \right) \mathrm{e}^x(\Gamma(f,f)(x_t)\mathcal{I}_{x_t\in D})\end{aligned}$$

for every $t \geq t_1$.

Proof. Consider the semigroup $P_t f(x) = \mathrm{e}^x f(x_t)$. Since $\frac{d}{ds} P_s = \mathcal{L} P_s = P_s \mathcal{L}$, we can calculate

$$P_t f^2(x) - (P_t f(x))^2 = \int_0^t \frac{d}{ds} P_s(P_{t-s} f)^2(x) ds = \int_0^t P_s \Gamma(P_{t-s} f, P_{t-s} f)(x) ds. \quad (9)$$

We want to bound $\Gamma(P_{t-s} f, P_{t-s} f)$ by $P_{t-s}\Gamma(f,f)$ so that the energy of the Poincaré inequality will be formed. If the process is such that the translation property $\mathrm{e}^{x+y} f(z) = \mathrm{e}^x f(z+y)$ holds, as in Wang & Yuan (2010); Ane & Ledoux (2000), then one can obtain the desired bound as shown below.

$$\begin{aligned}\Gamma(P_{t-s} f, P_{t-s} f)(x) = & \ \frac{1}{2} \sum_{i=1}^{N} \phi(x^i) (\mathrm{e}^{\Delta_i(x)} f(x_{t-s}) - \mathrm{e}^x f(x_{t-s}))^2 \\ = & \ \frac{1}{2} \sum_{i=1}^{N} \phi(x^i) (\mathrm{e}^x f(\Delta_i(x_{t-s})) - \mathrm{e}^x f(x_{t-s}))^2 \leq P_{t-s}\Gamma(f,f)(x).\end{aligned}$$

In our case where we do not have the translation property we will use a bound based on the Dynkin's formula

$$\mathrm{e}^y f(x_t) = f(y) + \int_0^t \mathrm{e}^y (\mathcal{L} f(x_u)) du,$$

we can consequently bound

$$\left(e^{\Delta_i(x)} f(x_{t-s}) - e^{x} f(x_{t-s})\right)^2 \leq 2\left(f(\Delta_i(x)) - f(x)\right)^2 + \\ + 2\left(\int_0^{t-s} \left(e^{\Delta_i(x)}(\mathcal{L}f(x_u)) - e^{x}(\mathcal{L}f(x_u))\right) du\right)^2.$$

To bound the second term we apply Lemma 3

$$\left(e^{\Delta_i(x)} f(x_{t-s}) - e^{x} f(x_{t-s})\right)^2 \leq 2\left(f(\Delta_i(x)) - f(x)\right)^2 \\ + 8\theta^2 t^2 M e^{x}(\Gamma(f,f)(x_t)\mathcal{I}_{x_t \in D}).$$

By the definition of the carré du champ we then get

$$\Gamma(P_{t-s}f, P_{t-s}f)(x) \leq 2\Gamma(f,f)(x) + 8\theta^2 t^2 M\left(\sum_{i=1}^{N}\phi(x^i)\right) e^{x}(\Gamma(f,f)(x_t)\mathcal{I}_{x_t \in D}).$$

If we combine the last one together with (9) we obtain

$$P_t f^2(x) - (P_t f(x))^2 \leq 2t\Gamma(f,f)(x) + 8\theta^2 t^3 M\left(\sum_{i=1}^{N}\phi(x^i)\right) e^{x}(\Gamma(f,f)(x_t)\mathcal{I}_{x_t \in D}).$$

From the last Lemma we obtain the following local Poincaré inequality.

Corollary 3. *For the PJMP as described in (1)-(4), we have*

$$\mu(f^2\mathcal{I}_D) \leq \mu((e^{x}(f(x_t)\mathcal{I}_{x_t \in D}))^2) + a_1(t)\mu(\Gamma((f,f))(x)) + \\ a_2(t)\mu\left(\left(\sum_{i=1}^{N}\phi(x^i)\right) e^{x}(\Gamma(f,f)(x_t)\mathcal{I}_{x_t \in D})\right).$$

Where $a_1(t) = 2t$ and $a_2(t) = 8\theta^3 t^3 M$.

Proof. Since for μ the invariant measure of P_t one has $\mu(x) = \sum_y \mu(y)P_t(y,x)$ we can write

$$\mu(f^2\mathcal{I}_D) = \sum_{x \in D}\mu(x)f^2(x) = \sum_{x \in D}\sum_{y}\mu(y)P_t(y,x)f^2(x) = \\ = \sum_{y}\mu(y)\sum_{x \in D}P_t(y,x)f^2(x). \tag{10}$$

If we now use Lemma 4 to bound the semigroup we obtain

$$\mu(f^2\mathcal{I}_D) \leq \mu((e^{x}(f(x_t)\mathcal{I}_{x_t \in D}))^2) + 2t\mu(\Gamma((f,f))(x)) + \\ 8\theta^3 t^3 M\mu\left(\left(\sum_{i=1}^{N}\phi(x^i)\right) e^{x}(\Gamma(f,f)(x_t)\mathcal{I}_{x_t \in D})\right).$$

Since we have already obtained local Poincaré inequalities, as well as the Lyapunov inequality required, in the following proposition we show how the two conditions, the local Poincaré of Corollary 3 and the Lyapunov inequality of Lemma 1, are sufficient for the Poincaré type inequality of Theorem 1.

Proposition 1. *Assume that for some* $V \geq 1$ *the Lyapunov inequality*

$$\mathcal{L}V \leq -\vartheta V + b\mathcal{I}_B$$

holds and that for some $D \supset B$ *we have the weighted local Poincaré*

$$\begin{aligned}\mu(f^2\mathcal{I}_D) \leq & \mu((\mathrm{e}^x(f(x_t)\mathcal{I}_{x_t\in D}))^2) + a_1(t)\mu(\Gamma((f,f))(x)\mathcal{I}_{x\in D}) + \\ & a_2(t)\mu\left(B(x)\mathrm{e}^x(\Gamma(f,f)(x_t)\mathcal{I}_{x_t\in D})\right) \end{aligned} \tag{11}$$

where $B(x)$ *a function of the initial configuration x. Then*

$$\int Var_{E^x}(f(x_t))d\mu \leq \delta(t)\mu(\Gamma(f,f)(x)) + a_2(t)\mu\left(B(x)P_t(\Gamma(f,f)\mathcal{I}_{x_t\in D})(x)\right)$$

where $\delta(t) = a_1(t) + \frac{d_1}{2\theta}$*, for some* $d_1 > 0$.

Proof. At first, we can write

$$\mu(f^2) = \mu(f^2\mathcal{I}_D) + \frac{1}{\vartheta}\mu(f^2\vartheta\mathcal{I}_{D^c}).$$

For the first term on the right hand side we can use (11), while for the second we can use the Lyapunov Inequality. That gives

$$\mu(f^2\vartheta\mathcal{I}_{D^c}) \leq \mu(f^2\frac{-\mathcal{L}V}{V}\mathcal{I}_{D^c}) + b\mu(f^2\mathcal{I}_{B\cap D^c}).$$

If we choose D large enough to contain the set B, i.e. $B \cap D^c = \emptyset$ the last one is reduced to

$$\mu(f^2\vartheta\mathcal{I}_{D^c}) \leq \mu(f^2\frac{-\mathcal{L}V}{V}\mathcal{I}_{D^c}).$$

The need to bound the quantity $\frac{-\mathcal{L}V}{V}$ which appears from the use of the Lyapunov inequality is the actual reason why we need to make use of the invariant measure μ and obtain the type of Poincaré inequality shown in our final result. If we had not taken the expectation with respect to the invariant measure, we would had to bound

$$\int f^2\frac{-\mathcal{L}V}{V}\mathcal{I}_{D^c}dP_t$$

instead. This, in the case of diffusions can be bounded by the carré du champ of the function $\Gamma(f,f)$ by making an appropriate selection of exponential decreasing density (see for instance Bakry et al. (2008, 2014); Cattiaux et al. (2009)). In the case of jump processes however, and in particular of PJMP as on the current paper where densities cannot been specified, a similar bound cannot be obtained. However, when it comes to the analogue expression involving the invariant measure there is a powerful result that we can use, which has been presented in Cattiaux et al. (2009) (see Lemma 2.12). According to this, if we take the expectation with respect to the invariant measure, the desired bound holds as seen in the following Lemma.

Lemma 5. *(Lemma 2.12 in Cattiaux et al. (2009)) For every* $U \geq 1$ *such that* $-\frac{\mathcal{L}U}{U}$ *is bounded from below, the following bound holds*

$$\mu f^2 \frac{-\mathcal{L}U}{U} \leq d_1 \mu(f(-\mathcal{L})f)$$

where μ *is the invariant measure of the process and* d_1 *is some positive constant.*

Since $V \geq 1$ and for $x \in D$ we have from the Lyapunov inequality that $-\frac{\mathcal{L}V}{V} \geq \vartheta$ we get the following bound

$$\mu(f^2 \vartheta \mathcal{I}_{D^c}) \leq d_1 \mu(f(-\mathcal{L})f)$$

for some positive constant d_1. Since for the infinitesimal operator $\mu(\mathcal{L}f) = 0$ for every function f, we can write

$$\int (f(-\mathcal{L})f) d\mu = \frac{1}{2} \int (\mathcal{L}(f^2) - 2f\mathcal{L}f) d\mu = \frac{1}{2} \int \Gamma(f,f) d\mu.$$

So that,

$$\mu(f^2 \theta \mathcal{I}_{D^c}) \leq \frac{d_1}{2} \mu(\Gamma(f,f)). \tag{12}$$

Gathering all together we finally obtain the desired inequality

$$\int f^2 d\mu \leq (a_1(t) + \frac{d_1}{2\vartheta}) \mu(\Gamma(f,f)(x)) + a_2(t) \mu\left(B(x) P_t(\Gamma(f,f)\mathcal{I}_{x_t \in D})(x)\right) + \mu((\mathrm{e}^x(f(x_t)\mathcal{I}_{x_t \in D}))^2)$$

which proves the proposition for a constant $\delta(t) = a_1(t) + \frac{d_1}{2\vartheta}$.

The last proposition together with the Lyapunov inequality from Lemma 1 and the local Poincaré inequality of Corollary 3 proves Theorem 1.

3.2 Proof of the Poincaré inequalities for the invariant measure

We start this section by proving a Poincaré inequality for the invariant measure. For this we will use Lyapunov methods developed in Bakry et al. (2008), Bakry et al. (2014) and Cattiaux et al. (2009).

Proposition 2. *For the PJMP as described in (1)-(4), assume that for some $V \geq 1$ the Lyapunov inequality*

$$\mathcal{L}V \leq -\vartheta V + b\mathcal{I}_B$$

holds. Then μ satisfies a Poincaré inequality

$$\mu\,(f - \mu f)^2 \leq C_0 \mu(\Gamma(f,f)),$$

for some $C_0 > 0$.

Proof. At first assume $\mu(f\mathcal{I}_D) = 0$. We can write

$$Var_\mu(f) = \int f^2 d\mu - \left(\int f\mathcal{I}_{D^c} d_\mu\right)^2 \leq \int f^2 \mathcal{I}_D d\mu + \frac{1}{\vartheta}\int f^2 \vartheta\, \mathcal{I}_{D^c} d\mu.$$

For the second term if we work as in Proposition 1, with the use of the Lyapunov inequality we have the following bound

$$\int f^2 \vartheta \mathcal{I}_{D^c} d\mu \leq \int f^2 \frac{-\mathcal{L}V}{V} \mathcal{I}_{D^c} d\mu \leq d_1 \int \Gamma(f,f)\mathcal{I}_{D^c} d\mu.$$

For the first term, we will use the approach applied in Saloff (1996) in order to prove Poincaré inequalities for finite Markov chains. Since we have assumed $\int f\mathcal{I}_D d\mu = 0$, we can write

$$\int f^2 \mathcal{I}_D d\mu = \frac{1}{2}\int\int (f(x) - f(y))^2\, \mathcal{I}_{x\in D}\mathcal{I}_{y\in D}\mu(dx)\mu(dy).$$

If we consider $J_{xy} = \left\{J_1, ..., J_{\|J_{xy}\|}\right\}$ to be the shortest sequence of spikes that leads from the configuration x to the configuration y without leaving D, then we can denote $\tilde{x}^0 = x$ and for every $k = 0, ..., \|J_{xy}\|$, $\tilde{x}^k = \Delta_{J_k}(\Delta_{J_{k-1}}(...\Delta_{J_1}(x))...)$, the configuration after the $k$the neuron on the sequence has spiked. Since D is finite, the length of the sequence is always uniformly bounded for any couple $x, y \in D$. Then

$$\mu(x)\mu(y)(f(x) - f(y))^2 \leq \mu(y)\mu(x)\sum_{j=0}^{|J_{xy}|}(f(\Delta(\tilde{x}^j)_{J_j}) - f(\tilde{x}^j))^2.$$

Since $\phi \geq \delta$ we have

$$\mu(x)\mu(y)(f(x) - f(y))^2 \leq \frac{\mu(y)\mu(x)}{\delta}\sum_{j=0}^{|J_{xy}|}\varphi(\tilde{x}^i_{J_j})(f(\Delta(\tilde{x}^j)_{J_j}) - f(\tilde{x}^j))^2.$$

If we form the carré du champ, we will obtain

$$\mu(x)\mu(y)(f(x)-f(y))^2 \leq \frac{\mu(y)\mu(x)}{\delta}\sum_{j=0}^{|J_{xy}|}\sum_{i\in D}\varphi(\tilde{x}^i_{J_j})(f(\Delta(\tilde{x}^j)_{J_j})-f(\tilde{x}^j))^2$$
$$\leq \frac{\mu(y)\mu(x)}{\text{mín}\{x\in D:\mu(x)\}\delta}\sum_{j=0}^{|J_{xy}|}\mu((\tilde{x}^j))\Gamma(f,f)(\tilde{x}^j).$$

This leads to

$$\int f^2\mathcal{I}_D d\mu \leq \frac{N^2}{2\,\text{mín}\{x\in D:\mu(x)\}\delta}\sum_{x\in D}\pi(x)\Gamma(f,f)(x)$$
$$=\frac{N^2}{2\,\text{mín}\{x\in D:\mu(x)\}\delta}\mu(\Gamma(f,f)\mathcal{I}_D).$$

Gathering everything together gives

$$Var_\mu(f)\leq\left(\frac{N^2}{2\,\text{mín}\{x\in D:\mu(x)\}\delta}+d_1\right)\int\Gamma(f,f)d\mu.$$

4 Proof of Talagrand inequality for the invariant measure

In the current section we prove concentration properties. At first we present the general proposition that connects the Poincaré inequality of Theorem 2 with measure concentration properties. These properties will be based on the following proposition, that follows closely the approach in Ledoux (2001) (see also Ledoux (1999), Bobkov & Ledoux (1997), Aida, Masuda, & Shigekawa (1994) and Aida & Stroock (1994)). We will also use elements from Barthe & Roberto (2008) since one of the main conditions (13), will refer to the bounded function $F_r=\text{mín}\{F,r\}$.

Proposition 3. *Assume that the Poincaré inequality*

$$Var_\mu(f)\leq C_0\int\Gamma(f,f)d\mu$$

holds, and that for some λ such that $\lambda<\frac{1}{\sqrt{C_0C_3}}$

$$\mu(\Gamma(e^{\lambda F_r/2},e^{\lambda F_r/2}))\leq\lambda^2C_3\mu\left(e^{\lambda(t)F_r}\right). \tag{13}$$

Then the following concentration inequality holds

$$\mu(\{F>r\})\leq\lambda_0e^{\lambda\mu(F)}e^{-\lambda r},$$

for some $\lambda_0>0$. Furthermore,

$$\mu(\{|P_tF-\mu f|>r\})\leq\lambda_0e^{\lambda\mu(F)}e^{-\lambda r}.$$

Proof. From the Poincaré inequality, for $f = e^{\frac{\lambda F_r}{2}}$, we have

$$\mu(e^{\lambda F_r}) \leq C_0 \mu(\Gamma(e^{\frac{\lambda}{2}F_r}, e^{\frac{\lambda}{2}F_r})) + \left(\mu e^{\frac{\lambda}{2}F_r}\right)^2 .$$

If we bound the carré du champ from condition (13)

$$\mu(e^{\lambda F_r}) \leq C_0 \lambda^2 C_3 \mu\left(e^{\lambda F_r}\right) + \left(\mu e^{\frac{\lambda}{2}F_r}\right)^2 .$$

For $\lambda < \frac{1}{\sqrt{C_0 C_3}}$ we get

$$\mu(e^{\lambda F_r}) \leq \frac{1}{1-\lambda^2 C_0 C_3}\left(\mu e^{\frac{\lambda}{2}F_r}\right)^2 .$$

Iterating this gives

$$\mu(e^{\lambda F_r}) \leq \prod_{k=0}^{n-1}\left(\frac{1}{1-\frac{\lambda^2 C_0 C_3}{4^k}}\right)^{2^k}\left(\mu(e^{\frac{\lambda}{2^n}F_r})\right)^{2^n} .$$

We notice that $\left(\mu(e^{\frac{\lambda}{2^n}F_r})\right)^{2^n} \to e^{\lambda\mu(F_r)}$ as $n \to \infty$ and that $\lambda_0 := \prod_{k=0}^{n-1}\left(\frac{1}{1-\frac{\lambda^2 C_0 C_3}{4^k}}\right)^{2^k} < \infty$ for $\lambda < \frac{1}{\sqrt{C_0 C_3}}$. So we get

$$\mu(e^{\lambda F_r}) \leq \lambda_0 e^{\lambda\mu(F_r)} < \infty.$$

Since

$$\{P_t F_r < r\} = \{P_t F < r\}$$

we can apply Chebyshev's inequality

$$\mu\left(\{P_t F > r\}\right) \leq e^{-\lambda r}\mu(e^{\lambda P_t F_r}) \leq e^{-\lambda r}\mu(P_t e^{\lambda F_r}) = e^{-\lambda r}\mu(e^{\lambda F_r}),$$

because of Jensen's inequality and the invariant measure property $\mu P_t = \mu$. Similarly, since

$$\{F_r < r\} = \{F < r\}$$

we also have

$$\mu\left(\{F > r\}\right) \leq e^{-\lambda r}\mu(e^{\lambda F_r}).$$

To get the final result, at first we substitute F with $F-\mu(F)$. Then repeat the same for $-F$ and the result follows.

To complete the proofs of concentration Theorems 3 and 4 and of Corollary 3, we need to verify (13). We start with Theorem 4. We have to show condition (13) for $F(x) = \sum_{i=1}^N x^i$. This is shown in next Lemma.

Lemma 6. *Assume the PJMP as described in (1)-(4). Let* $F(x) = \sum_{i=1}^N x^i$ *for* $x = (x^1, ..., x^N) \in \mathbb{R}_+^N$. *Then*

$$\mu(\Gamma(e^{\lambda F_r/2}, e^{\lambda F_r/2})) \leq C_3 \lambda^2 \mu\left(e^{\lambda F_r}\right)$$

where $F_r = \text{mín}(F(x), r)$ *for* $r > 0$*.*

Proof. Using the definition of $\Gamma(,)$

$$\mu(\Gamma(e^{\lambda F_r/2}, e^{\lambda F_r/2})) = \sum_{i=1}^{N} \mu\left(\underbrace{\phi(x^i)(e^{\lambda F_r(x)/2} - e^{\lambda F_r(\varDelta_i(x))/2})^2}_{M_i}\right).$$

To bound $\mu(M_i)$ we will distinguish four cases:

a) Consider the set $A := \{x : F(x) \geq r \text{ and } F(\varDelta_i(x)) \geq r\}$. Then, for $x \in A$ $F_r(\varDelta_i(x)) = F_r(x) = r$ and so $\mu(M_i \mathcal{I}_A) = 0$.

b) Consider the set $B := \{x : F(x) \geq r \text{ and } F(\varDelta_i(x)) \leq r\}$. Then, for $x \in B$,

$$F_r(\varDelta_i(x)) = \sum_{j, j \neq i} \varDelta_i(x)^j < r = F_r(x) \leq \sum_j x^j$$

so that

$$\mu(M_i \mathcal{I}_B) \leq \lambda^2 \mu\left(\phi(x^i) e^{\lambda F_r(x)} (F_r(x) - F_r(\varDelta_i(x)))^2\right) \leq$$
$$\leq \lambda^2 e^{\lambda r} \mu\left(\phi(x^i)(F_r(x) - F_r(\varDelta_i(x)))^2\right).$$

Since $F_r(\varDelta_i(x)) = \sum_{j,j\neq i} W_{i\to j} + \sum_{j,j\neq i} x^j < r \leq \sum_j x^j$ we have

$$F_r(x) - F_r(\varDelta_i(x)) = r - \left(\sum_{j,j\neq i} x^j + \sum_{j,j\neq i} W_{i\to j}\right) < x^i - \sum_{j,j\neq i} W_{i\to j}$$

which leads to

$$\mu(M_i \mathcal{I}_B) \leq \lambda^2 e^{\lambda r} \mu\left(\phi(x^i)(x^i - \sum_{j,j\neq i} W_{i\to j})^2\right) = C^i \lambda^2 e^{\lambda r} = C^i \lambda^2 \mu(e^{\lambda F_r} \mathcal{I}_B)$$

where above we denoted $C^i = \mu(\phi(x^i)(x^i)^2) + N_0^2 \mu(\phi(x^i))$ and computed

$$e^{\lambda F_r} \mathcal{I}_B = e^{\lambda r} \mathcal{I}_B = \mu(e^{\lambda r} \mathcal{I}_B) = \mu(e^{\lambda F_r} \mathcal{I}_B).$$

c) Consider the set $C := \{F(\varDelta_i(x)) \leq F(x) < r\}$. Then, for $x \in C$,

$$F_r(x) - F_r(\varDelta_i(x)) = \sum_j x^j - \left(\sum_{j,j\neq i} x^j + \sum_{j,j\neq i} W_{i\to j}\right) = x^i - \sum_{j,j\neq i} W_{i\to j} \geq 0,$$

so that

$$\begin{aligned}\mu(M_i\mathcal{I}_C) \leq& \lambda^2\mu\left(\phi(x^i)e^{\lambda F_r(x)}(F_r(x)-F_r(\Delta_i(x)))^2\right)\\ \leq&\lambda^2\mu\left(\phi(x^i)e^{\lambda F_r(x)}(x^i-\sum_{j,j\neq i}W_{i\to j})^2\right)\\ \leq&\lambda^2\mu\left(\phi(x^i)e^{\lambda F_r(x)}(x^i)^2\right).\end{aligned}$$

Since $F_r \leq r$ we know that $\mu(e^{\lambda F_r}) \leq e^{\lambda r} < \infty$ and so we can bound

$$\mu(M_i\mathcal{I}_C) \leq \lambda^2(\sup_{g:\mu(g)=1}\{\phi(x^i)(x^i)^2g\})\mu\left(e^{\lambda F_r}\mathcal{I}_C\right) \leq \lambda^2|||\phi(x^i)(x^i)^2|||_\infty\mu\left(e^{\lambda F_r}\mathcal{I}_C\right)$$

where

$$|||f|||_\infty = \sup_{g:\mu(g)=1}\{\mu(fg)\}.$$

d) Consider the set $D := \{F(x) < r \text{ and } F(x) < F(\Delta_i(x))\}$. Then, for $x \in D$,

$$\sum_j x^j = F_r(x) < F_r(\Delta_i(x)) \leq \sum_{j,j\neq i}W_{i\to j} + \sum_{j,j\neq i}x^j = F_r(x) + \left(\sum_{j,j\neq i}W_{i\to j} - x^i\right)$$

which means that x^i is bounded by

$$x^i \leq \sum_{j,j\neq i}W_{i\to j} \leq N_0$$

and that

$$0 \leq F_r(\Delta_i(x)) - F_r(x) \leq \sum_{j,j\neq i}W_{i\to j} - x^i.$$

So, we can compute

$$\begin{aligned}\mu(M_i\mathcal{I}_D) \leq&\lambda^2\mu\left(\phi(x^i)e^{\lambda F_r(x)}(F_r(x)-F_r(\Delta_i(x)))^2\right)\\ \leq&\lambda^2\mu\left(\phi(x^i)e^{\lambda F_r(\Delta_i(x))}(x^i-\sum_{j,j\neq i}W_{i\to j})^2\right)\\ \leq&\lambda^2\mu\left(\phi(x^i)e^{\lambda F_r(x)}e^{\lambda(\sum_{j,j\neq i}W_{i\to j}-x^i)}(x^i-\sum_{j,j\neq i}W_{i\to j})^2\right)\\ \leq&N_0^2\phi(N_0)\lambda^2e^{\lambda N_0}\mu\left(e^{\lambda F_r}\mathcal{I}_D\right).\end{aligned}$$

If we gather all four cases together, we finally obtain

$$\mu(\Gamma(e^{\lambda F_R/2}, e^{\lambda F_r/2})) \leq C\lambda^2\mu\left(e^{\lambda F_r}\right)$$

for a constant

$$C_3 = \text{máx}\{\mu(\phi(x^i)(x^i)^2) + N_0^2\mu(\phi(x^i)), |||\phi(x^i)(x^i)^2|||_\infty, N_0^2\phi(N_0)e^{\lambda N_0}\}.$$

In the remaining of the paper, we prove the concentration properties presented in Theorem 3 and Corollary 2. What remains is to present conditions so that (13) of Proposition 3 holds.

As one can see in the main tool to show concentration properties presented in Proposition 3, we need to bound $\mu(\Gamma(e^{\lambda f/2}, e^{\lambda f/2}))$. In the case of diffusion, where $\mu(\Gamma(f,f)) = \mu(\|\nabla f\|^2)$, for any smooth function ψ one has

$$\mu(\Gamma(\psi(f),\psi(f))) \leq \|\nabla f\|_\infty^2 \mu(\psi'(f)^2)$$

and so one can bound $\mu(\Gamma(e^{\lambda f/2}, e^{\lambda f/2})) \leq \frac{\lambda^2}{4}\|\nabla f\|_\infty^2 \mu(e^{\lambda f})$, and so the condition follows for functions f such that $\|\nabla f\|_\infty^2 < 1$ (see Ledoux (2001) and Ledoux (1999)). In the case, as is in the current paper, of an energy expressed through differences, where the chain rule is not satisfied, this cannot hold. However, as demonstrated in Aida & Stroock (1994) (see also Gross & Rothaus (1998) for applications), in the special situation where the semigroup is symmetric, one can have an analogue result, that is

$$\mu(\Gamma(e^{\lambda f/2}, e^{\lambda f/2})) \leq \frac{\lambda^2}{4} |||| f ||||_\infty^2 \mu(e^{\lambda f(x)})$$

where now $||||f||||_\infty$ can be considered as a generalised norm of the gradient (see also Ledoux (2001)), given by the following expression

$$||||f||||_\infty = \sup\left\{\mathcal{E}(gf,f) - \frac{1}{2}\mathcal{E}(g,f^2); g : \|g\|_1 \leq 1\right\}$$

where $\mathcal{E}(f,g) := \lim_{t\to 0}\frac{1}{2t}\int\int (f(x)-f(y))^2 p_t(x,dy)\mu(dx)$. Then, of course, for the concentration property to hold, one needs functions that satisfy the following condition $||||f||||_\infty < 1$. In our case however, we can still obtain the desired property for a different class of functions, that satisfy

$$|||\phi(x^i)D(f)^2|||_\infty < 1 \text{ and } |||\phi(x^i)e^{\lambda D(f)}D(f)^2|||_\infty < 1$$

where $|||f|||_\infty = \sup_{g:\mu(g)=1}\{\mu(fg)\})$.

In the following Lemma we show condition (13) under the hypothesis $|||\phi(x^i)D(f)^2|||_\infty < 1$ and $|||\phi(x^i)e^{\lambda D(f)}D(f)^2|||_\infty < 1$ of Theorem 3, for non-compact neurons as in (4)-(1).

Lemma 7. *Assume the PJMP as described in (1)-(4). Assume functions f such that*

$$|||\phi(x^i)D(f)^2|||_\infty < 1 \textit{ and } |||\phi(x^i)e^{\lambda D(f)}D(f)^2|||_\infty < 1,$$

where $D(f) = \sup_{i=1}^N |f(x) - f(\Delta_i(x))|$. Then

$$\mu(\Gamma(e^{\lambda f_r/2}, e^{\lambda f_r/2})) \leq C_3\lambda^2\mu\left(e^{\lambda f_r}\right).$$

Proof. Using the definition of $\Gamma(,)$ we compute

$$\mu(\Gamma(e^{\lambda f_r/2}, e^{\lambda f_r/2})) = \sum_{i=1}^{N} \mu\left(\underbrace{\phi(x^i)(e^{\lambda f_r(x)/2} - e^{\lambda f_r(\Delta_i(x))/2})^2}_{M_i}\right).$$

a) Consider the set $A := \{x : f(x) \geq r \text{ and } f(\Delta_i(x)) \geq r\}$. Then, for $x \in A$, $f_r(\Delta_i(x)) = f_r(x) = r$ and so $\mu(M_i \mathcal{I}_A) = 0$.

b) Consider the set $B := \{x : f(x) \geq r \text{ and } f(\Delta_i(x)) \leq r\}$. Then, for $x \in B$,

$$\mu(M_i \mathcal{I}_B) \leq \lambda^2 \mu\left(\phi(x^i) e^{\lambda f_r(x)} D(f)^2\right) \leq \lambda^2 e^{\lambda r} \mu\left(\phi(x^i) D(f)^2\right)$$

which leads to

$$\mu(M_i \mathcal{I}_B) \leq \mu\left(\phi(x^i) D(f)^2\right) \lambda^2 \mu(e^{\lambda f_r} \mathcal{I}_B)$$

since

$$e^{\lambda f_r} \mathcal{I}_B = e^{\lambda r} \mathcal{I}_B = \mu(e^{\lambda r} \mathcal{I}_B) = \mu(e^{\lambda f_r} \mathcal{I}_B).$$

c) Consider the set $C := \{f(\Delta_i(x)) \leq f(x) < r\}$. Then, for $x \in C$,

$$\begin{aligned}\mu(M_i \mathcal{I}_C) \leq& \lambda^2 \mu\left(\phi(x^i) e^{\lambda f_r(x)} (f_r(x) - f_r(\Delta_i(x)))^2\right) \\ \leq& \lambda^2 \mu\left(\phi(x^i) e^{\lambda f_r(x)} D(f)^2\right).\end{aligned}$$

Since $f_r \leq r$ we know that $\mu(e^{\lambda f_r}) \leq e^{\lambda r} < \infty$ and so we can bound

$$\mu(M_i \mathcal{I}_C) \leq \lambda^2 (\sup_{g: \mu(g)=1} \{\phi(x^i) D(f)^2 g\}) \mu\left(e^{\lambda f_r} \mathcal{I}_C\right) \leq \lambda^2 |||\phi(x^i) D(f)^2|||_\infty \mu\left(e^{\lambda f_r} \mathcal{I}_C\right)$$

where

$$|||f|||_\infty = \sup\{\mu(fg);\ g : \mu(g) \leq 1\}.$$

d) Consider the set $D := \{f(x) < r \text{ and } f(x) < f(\Delta_i(x))\}$. Then, for $x \in D$,

$$\begin{aligned}\mu(M_i \mathcal{I}_D) \leq& \lambda^2 \mu\left(\phi(x^i) e^{\lambda f_r(\Delta_i(x))} (f_r(x) - f_r(\Delta_i(x)))^2\right) \\ \leq& \lambda^2 \mu\left(\phi(x^i) e^{\lambda f_r(x)} e^{\lambda D(f)} D(f)^2\right) \\ \leq& \lambda^2 |||\phi(x^i) e^{\lambda D(f)} D(f)^2|||_\infty \mu\left(e^{\lambda f_r} \mathcal{I}_D\right),\end{aligned}$$

where again we used that $e^{\lambda f_r} \mathcal{I}_D = e^{\lambda r} \mathcal{I}_D = \mu(e^{\lambda r} \mathcal{I}_D) = \mu(e^{\lambda f_r} \mathcal{I}_D)$.

We then have

$$\begin{aligned}\mu(\Gamma(e^{\lambda f_r/2}, e^{\lambda f_r/2})) \leq& \lambda^2 \sum_{i=1}^{N} \left(2|||\phi(x^i) D(f)^2|||_\infty + |||\phi(x^i) e^{\lambda D(f)} D(f)^2|||_\infty\right) \mu\left(e^{\lambda f_r}\right) \\ \leq& 3N \lambda^2 \mu\left(e^{\lambda f_r}\right),\end{aligned}$$

and the Lemma follows for some constant $C_3 = 3N$.

5 Conclusion

In the current work we studied Poincaré and Talagrand concentration inequalities for neurons with noncompact membrane potential. In both the Poincaré inequality and the Talagrand inequality, the main tool to control the behaviour of the neurons when their membrane potential takes big values was a Lyapunov inequality

$$\mathcal{L}V \leq -\vartheta V + b\mathcal{I}_B$$

for some compact B. In order to satisfy this property we require the intensity function ϕ to be strictly bigger than zero (1) and increase fast enough (2). The Poincaré inequality presented in Theorem 2 and the concentration properties of Theorems 3 and 4 imply that whenever the membrane potential of the neurons gets high values away from some compact set, the system returns exponentially fast back to the compact set.

6 Bibliography

References

Aida, S., Masuda, T., & Shigekawa, I. (1994). Logarithmic Sobolev inequalities and exponential integrability. *Journal of Functional Analysis*, *126*(1), 83–101.

Aida, S., & Stroock, D. (1994). Moment estimates derived from Poincaré and logarithmic Sobolev inequalities. *Mathematical Research Letters*, *1*(1), 75–86.

André, M. (2019). A result of metastability for an infinite system of spiking neurons. *Journal of Statistical Physics*, *177*(5), 984–1008.

André, M., & Planche, L. (2021). The effect of graph connectivity on metastability in a stochastic system of spiking neurons. *Stochastic Processes and their Applications*, *131*, 292–310.

Ane, C., & Ledoux, M. (2000). Rate of convergence for ergodic continuous Markov processes: Lyapunov versus Poincaré. *Probab. Theory Relat. Fields*, *116*, 573–602.

Azaïs, R., Bardet, J.-B., Génadot, A., Krell, N., & Zitt, P.-A. (2014). Piecewise deterministic Markov process—recent results. En *Esaim: Proceedings* (Vol. 44, pp. 276–290).

Bakry, D., Cattiaux, P., & Guillin, A. (2008). Rate of convergence for ergodic continuous Markov processes: Lyapunov versus Poincaré. *Journal of Functional Analysis*, *254*(3), 727–759.

Bakry, D., Gentil, I., & Ledoux, M. (2014). *Analysis and geometry of Markov diffusion operators* (Vol. 103). Springer.

Barthe, F., & Roberto, C. (2008). Modified Logarithmic Sobolev Inequalities on $\backslash\mathbb{R}$. *Potential Analysis*, *29*(2), 167–193.

Bobkov, S., & Ledoux, M. (1997). Poincaré's inequalities and Talagrand's concentration phenomenon for the exponential distribution. *Probability Theory and Related Fields*, *107*(3), 383–400.

Cattiaux, P., Guillin, A., Wang, F.-Y., & Wu, L. (2009). Lyapunov conditions for super Poincaré inequalities. *Journal of Functional Analysis*, *256*(6), 1821–1841.

Chafaï, D. (2004). Entropies, convexity, and functional inequalities, On $\backslash\phi -$ entropies and$\backslash\phi -$Sobolev inequalities. *Journal of Mathematics of Kyoto University*, *44*(2), 325–363.

Chevallier, J. (2017). Mean-field limit of generalized Hawkes processes. *Stochastic Processes and their Applications*, *127*(12), 3870–3912.

Crudu, A., Debussche, A., Muller, A., & Radulescu, O. (2012). Convergence of stochastic gene networks to hybrid piecewise deterministic processes. *The Annals of Applied Probability*, *22*(5), 1822–1859.

Davis, M. H. (1984). Piecewise-deterministic Markov processes: A general class of non-diffusion stochastic models. *Journal of the Royal Statistical Society: Series B (Methodological)*, *46*(3), 353–376.

Davis, M. H. (1993). *Markov models and optimization. Monographs on Statistics and Applied Probability*. London: Chapman Hall.

Diaconis, P., & Saloff-Coste, L. (1996). Logarithmic Sobolev inequalities for finite Markov chains. *The Annals of Applied Probability*, *6*(3), 695–750.

Duarte, A., Löcherbach, E., & Ost, G. (2019). Stability, convergence to equilibrium and simulation of non-linear Hawkes processes with memory kernels given by the sum of Erlang kernels. *ESAIM: Probability and Statistics*, *23*, 770–796.

Duarte, A., & Ost, G. (2014). A model for neural activity in the absence of external stimuli. *arXiv preprint arXiv:1410.6086*.

Galves, A., & Löcherbach, E. (2013). Infinite systems of interacting chains with memory of variable length—a stochastic model for biological neural nets. *Journal of Statistical Physics*, *151*(5), 896–921.

Gentil, I., Guillin, A., & Miclo, L. (2005). Modified logarithmic Sobolev inequalities and transportation inequalities. *Probability theory and related fields*, *133*(3), 409–436.

Gross, L., & Rothaus, O. (1998). Herbst inequalities for supercontractive semigroups. *Journal of Mathematics of Kyoto University*, *38*(2), 295–318.

Guionnet, A., & Zegarlinksi, B. (2003). Lectures on logarithmic Sobolev inequalities. En *Séminaire de probabilités xxxvi* (pp. 1–134). Springer.

Hansen, N. R., Reynaud-Bouret, P., & Rivoirard, V. (2015). Lasso and probabilistic inequalities for multivariate point processes. *Bernoulli*, *21*(1), 83–143.

Hodara, P., Krell, N., & Löcherbach, E. (2016). Non-parametric estimation of the spiking rate in systems of interacting neurons. *Statistical Inference for Stochastic Processes*, *21*(1), 1–16.

Hodara, P., & Löcherbach, E. (2017). Hawkes processes with variable length memory and an infinite number of components. *Advances in Applied Probability*, *49*(1), 84–107.

Hodara, P., & Papageorgiou, I. (2019). Poincaré-Type Inequalities for Compact Degenerate Pure Jump Markov Processes. *Mathematics*, *7*(6), 518.

Inglis, J., & Papageorgiou, I. (2014). Log-Sobolev inequalities for infinite-dimensional Gibbs measures with non-quadratic interactions. *arXiv preprint arXiv:1401.3219*.

Ledoux, M. (1999). Concentration of measure and logarithmic Sobolev inequalities. En *Seminaire de probabilites xxxiii* (pp. 120–216). Springer.

Ledoux, M. (2001). *The concentration of measure phenomenon* (n.º 89). American Mathematical Soc.

Löcherbach, E. (2018). Absolute continuity of the invariant measure in piecewise deterministic Markov processes having degenerate jumps. *Stochastic Processes and their Applications*, *128*(6), 1797–1829.

Pakdaman, K., Thieullen, M., & Wainrib, G. (2010). Fluid limit theorems for stochastic hybrid systems with application to neuron models. *Advances in Applied Probability*, *42*(3), 761–794.

Papageorgiou, I. (2011). A note on the modified log-Sobolev inequality. *Potential Analysis*, *35*(3), 275–286.

Papageorgiou, I. (2020). Modified Log-Sobolev Inequality for a Compact Pure Jump Markov Process with Degenerate Jumps. *Journal of Statistical Physics*, *178*(6), 1293–1318.

Saloff, L. (1996). Coste. Lectures on finite Markov chains, volume 1665 of Lectures on Probability Theory and Statistics. *Lecture Notes in Math. Springer Verlag, Berlin*.

Talagrand, M. (1991). A new isoperimetric inequality and the concentration of measure phenomenon. En *Geometric aspects of functional analysis* (pp. 94–124). Springer.

Talagrand, M. (1995). Concentration of measure and isoperimetric inequalities in product spaces. *Publications Mathématiques de l'Institut des Hautes Etudes Scientifiques*, *81*(1), 73–205.

Wang, F.-Y., & Yuan, C. (2010). Poincaré inequality on the path space of Poisson point processes. *Journal of Theoretical Probability*, *23*(3), 824–833.