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Apollonius' problem using equations of tangent circles

El problema de Apolonio usando ecuaciones de círculos tangentes

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Abstract Tangent conic sections to the graph of a function are used to solve the Apollonius' problem. This manuscript presents a new method to solve this problem. The statement of the Apollonius' problem can originate ten cases. Here, three cases are solved. Namely, three lines (LLL), one line and two points (LPP) and three circles (CCC). These three combinations consider the three objects: circle, line and point. The solution is similar in the other seven cases of the problem. When the objects, line or circle, are part of the elements of the problem, the line or circle are taken as functions. When a point is an element of the problem, the equation of the tangent circle must contain this point. The equations of the tangent circles in the form center-radius are applied to these functions. Since the unknown tangent circle is tangent to the other objects (or passes through the eventual given points) of the problem, the different equations produce a system of non-linear equations. From the solution of this system of equations can be obtained the center-radius of the unknown tangent circle and the points of tangency.
Keywords Apollonius, problem, tangent, circle.

Resumen Secciones cónicas tangentes a la gráfica de una función son usadas para resolver el problema de Apolonio. Este artículo presenta un nuevo método para resolver este problema. El planteamiento del problema de Apolonio puede originar diez casos. Aquí, se resuelven tres casos. A saber, tres líneas (LLL), una línea y dos puntos (LPP) y tres círculos (CCC). Estas tres combinaciones consideran los tres objetos: círculo, línea y punto. La solución es similar en los otros siete casos del problema. Cuando los objetos, línea o círculo, son parte de los elementos del problema, la línea o el círculo se toman como funciones. Cuando un punto es un elemento del problema, la ecuación

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del círculo tangente debe contener este punto. A estas funciones se les aplican las ecuaciones de los círculos tangentes en la forma centro-radio. Dado que el desconocido círculo tangente es tangente a los otros objetos (o pasa por los eventuales puntos dados) del problema, las diferentes ecuaciones producen un sistema de ecuaciones no lineales. De la solución de este sistema de ecuaciones se puede obtener el centro-radio del círculo tangente desconocido y los puntos de tangencia.
Palabras Claves Apolonio, problema, tangente, círculo.

1 Introduction

The Apollonius' problem is as follows: "To construct the circle o circles tangent to three given circles" (Johnson, 1960, p. 118) The statement of the Apollonius' problem can originate ten cases of the problem. The unknown circle can be tangent to any three objects (circles, lines and points). The most difficult case of the problem is the three circles (CCC). The three circles (CCC) problem has generally eight solutions. Various geometric and analytical solutions have been found for the Apollonius' problem (Altshiller-Court, 1952; Courant & Robbins, 1941; Coxeter, 1968; Gheorghe, 2020; Gisch & Ribando, 2004; Johnson, 1960; Lewis & Bridgett, 2003; Muirhead, 1896). Here, a new solution strategy to the previously published solutions is shown. The solution is based in the application of the tangent conic sections to the graph of a function (Gómez-Villarraga, 2021).

A tangent circle can be obtained from a tangent conic section. A tangent circle can be obtained from a tangent ellipse taking the semi-major axis and the semi-minor axis equal to the radius (Gómez-Villarraga, 2021, 2022; Leithold, 1998; Swokowski & Cole, 2008). The equations of tangent circles in the form centerradius are the most appropriate to solve the Apollonius' problem. The equations in the form center-radius are obtained from the parametric equations of tangent circles (Gómez-Villarraga, 2021). The solution to the Apollonius' problem is illustrated with the three cases: three lines (LLL), one line and two points (LPP) and three circles (CCC). The different combinations in the Apollonius' problem can originate ten cases of the problem (Altshiller-Court, 1952; Courant & Robbins, 1941; Coxeter, 1968; Johnson, 1960). The solution strategy is similar in the other seven cases of the problem.

The solution method is described next. The Apollonius' problem considers three objects: circle, line and point (Altshiller-Court, 1952; Courant & Robbins, 1941; Coxeter, 1968; Johnson, 1960). When the objects, line or circle, are part of the elements of the problem, the line or circle are taken as functions. When a point is an element of the problem, the equation of the tangent circle must contain this point. The equations of the tangent circles in the form center-radius are applied to these functions. At this point, the tangent circles to the functions are determined. Since the unknown tangent circle is tangent to the other objects (or passes through the eventual given points) of the problem, comparing the different equations produces a system of non-linear equations. The solution of this system of equations gives the radius of the unknown tangent circle and the points of tangency. The center of the tangent circle can be obtained from the previous results and any of the equations in the form center-radius.

Additional calculations are performed. The values of the derivatives at the points of tangency for the given elements (lines or circles) and the obtained tangent circle are calculated. The results here obtained (center-radius) for the case three circles (CCC) are compared with the results from other previously published method (Courant & Robbins, 1941, p. 125-127).

2 Equations of tangent circles in the form center-radius

The parametric equations for the first tangent circle[1] with radius r to the graph of a function f at the point $P\left(x_0, f\left(x_0\right)\right)$ are given by (Gómez-Villarraga, 2021):

$$x(t)=\frac{r\sin(t)-f'(x_0)r[\cos(t)+1]}{\sqrt{1+[f'(x_0)]^2}}+x_0 \text{ (Gómez-Villarraga, 2021, p. 38)} \quad (1)$$

$$y(t)=\frac{f'(x_0)r\sin(t)+r[\cos(t)+1]}{\sqrt{1+[f'(x_0)]^2}}+f\left(x_0\right) \text{ (Gómez-Villarraga, 2021, p. 38)} \quad (2)$$

Reordering the equations (1) and (2):

$$x(t)-\left[x_0-\frac{rf'\left(x_0\right)}{\sqrt{1+\left[f'\left(x_0\right)\right]^2}}\right]=\frac{r\sin(t)-rf'\left(x_0\right)\cos(t)}{\sqrt{1+\left[f'\left(x_0\right)\right]^2}} \quad (3)$$

$$y(t)-\left[f\left(x_0\right)+\frac{r}{\sqrt{1+\left[f'\left(x_0\right)\right]^2}}\right]=\frac{rf'\left(x_0\right)\sin(t)+r\cos(t)}{\sqrt{1+\left[f'\left(x_0\right)\right]^2}} \quad (4)$$

Adding the square of the equations (3) and (4)[2] (Appendix A.1):

$$\left\{x-\left[x_0-\frac{rf'(x_0)}{\sqrt{1+[f'(x_0)]^2}}\right]\right\}^2+\left\{y-\left[f\left(x_0\right)+\frac{r}{\sqrt{1+[f'(x_0)]^2}}\right]\right\}^2=r^2 \quad (5)$$

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[1] a and b have been replaced by r.

[2] Replacing $x(t)$ and $y(t)$ by x and y.

Similarly, the parametric equations for the second tangent circle[3] with radius r to the graph of a function f at the point $P\left(x_0, f\left(x_0\right)\right)$ are given by (Gómez-Villarraga, 2021):

$$x(t) = \frac{-r\sin(t)+f'\left(x_0\right)r[\cos(t)+1]}{\sqrt{1+\left[f\left(x_0\right)\right]^2}} + x_0 \text{ (Gómez-Villarraga, 2021, p. 38)} \tag{6}$$

$$y(t) = \frac{-f'\left(x_0\right)r\sin(t)-r[\cos(t)+1]}{\sqrt{1+\left[f'\left(x_0\right)\right]^2}} + f\left(x_0\right) \text{ (Gómez-Villarraga, 2021, p. 38)} \tag{7}$$

Reordering the equations (6) and (7) :

$$x(t) - \left[x_0 + \frac{rf'\left(x_0\right)}{\sqrt{1+\left[f'\left(x_0\right)\right]^2}}\right] = \frac{-r\sin(t) + rf'\left(x_0\right)\cos(t)}{\sqrt{1+\left[f'\left(x_0\right)\right]^2}} \tag{8}$$

$$y(t) - \left[f\left(x_0\right) - \frac{r}{\sqrt{1+\left[f'\left(x_0\right)\right]^2}}\right] = \frac{-rf'\left(x_0\right)\sin(t) - r\cos(t)}{\sqrt{1+\left[f'\left(x_0\right)\right]^2}} \tag{9}$$

Adding the square of the equations (8) and (9)[4] (Appendix A.2):

$$\left\{x - \left[x_0 + \frac{rf'\left(x_0\right)}{\sqrt{1+\left[f'\left(x_0\right)\right]^2}}\right]\right\}^2 + \left\{y - \left[f\left(x_0\right) - \frac{r}{\sqrt{1+\left[f'\left(x_0\right)\right]^2}}\right]\right\}^2 = r^2 \tag{10}$$

3 Apollonius' problem using equations of tangent circles

3.1 Three lines (LLL)

The circles tangent to the three lines l_1, l_2 and l_3 are found in this case. The lines l_1, l_2 and l_3 are shown in the figure 1a. The equations of the lines are taken as functions and their derivatives are calculated:

$$f(x) = y = x + 3 \text{ (line } l_1) \tag{11}$$
$$f'(x) = y' = 1 \tag{12}$$
$$f(x) = y = -x + 1 \text{ (line } l_2) \tag{13}$$
$$f'(x) = y' = -1 \tag{14}$$
$$f(x) = y = -5x + 18 \text{ (line } l_3) \tag{15}$$
$$f'(x) = y' = -5 \tag{16}$$

[3] a and b have been replaced by r.

[4] Replacing $x(t)$ and $y(t)$ by x and y.

First, the tangent circle $\mathbb{C}_1$ is calculated. $x_{\mathbb{C}_1 l_1}$ corresponds to the tangency point between the circle $\mathbb{C}_1$ and the line l_1. The equation (10) is used to determine the second tangent circle $\mathbb{C}_1$ in relation to the line l_1 :

$$\left\{x-\left[x_{\mathbb{C}_1 l_1}+\frac{\mathbb{R}_1 f'\left(x_{\mathbb{C}_1 l_1}\right)}{\sqrt{1+\left[f'\left(x_{\mathbb{C}_1 l_1}\right)\right]^2}}\right]\right\}^2+\left\{y-\left[f\left(x_{\mathbb{C}_1 l_1}\right)-\frac{\mathbb{R}_1}{\sqrt{1+\left[f'\left(x_{\mathbb{C}_1 l_1}\right)\right]^2}}\right]\right\}^2=\mathbb{R}_1^2 \quad (17)$$

The equations (11) and (12) are used to calculate $f\left(x_{\mathbb{C}_1 l_1}\right)$ and $f'\left(x_{\mathbb{C}_1 l_1}\right)$:

$$f\left(x_{\mathbb{C}_1 l_1}\right)=x_{\mathbb{C}_1 l_1}+3 \quad (18)$$

$$f'\left(x_{\mathbb{C}_1 l_1}\right)=1 \quad (19)$$

Replacing the equations (18) and (19) in the equation (17) :

$$\left\{x-\left[x_{\mathbb{C}_1 l_1}+\frac{\mathbb{R}_1}{\sqrt{2}}\right]\right\}^2+\left\{y-\left[x_{\mathbb{C}_1 l_1}+3-\frac{\mathbb{R}_1}{\sqrt{2}}\right]\right\}^2=\mathbb{R}_1^2 \quad (20)$$

The tangent circle $\mathbb{C}_1$ can also be calculated in relation to the lines l_2 and $l_3 \cdot x_{\mathbb{C}_1 l_2}$ corresponds to the tangency point between the circle $\mathbb{C}_1$ and the line l_2. The equation 5 is used to determine the first tangent circle $\mathbb{C}_1$ in relation to the line l_2 :

$$\left\{x-\left[x_{\mathbb{C}_1 l_2}-\frac{\mathbb{R}_1 f'\left(x_{\mathbb{C}_1 l_2}\right)}{\sqrt{1+\left[f'\left(x_{\mathbb{C}_1 l_2}\right)\right]^2}}\right]\right\}^2+\left\{y-\left[f\left(x_{\mathbb{C}_1 l_2}\right)+\frac{\mathbb{R}_1}{\sqrt{1+\left[f'\left(x_{\mathbb{C}_1 l_2}\right)\right]^2}}\right]\right\}^2=\mathbb{R}_1^2 \quad (21)$$

The equations (13) and (14) are used to calculate $f\left(x_{\mathbb{C}_1 l_2}\right)$ and $f'\left(x_{\mathbb{C}_1 l_2}\right)$:

$$f\left(x_{\mathbb{C}_1 l_2}\right)=-x_{\mathbb{C}_1 l_2}+1 \quad (22)$$

$$f'\left(x_{\mathbb{C}_1 l_2}\right)=-1 \quad (23)$$

Replacing the equations (22) and (23) in the equation (21) :

$$\left\{x-\left[x_{\mathbb{C}_1 l_2}+\frac{\mathbb{R}_1}{\sqrt{2}}\right]\right\}^2+\left\{y-\left[-x_{\mathbb{C}_1 l_2}+1+\frac{\mathbb{R}_1}{\sqrt{2}}\right]\right\}^2=\mathbb{R}_1^2 \quad (24)$$

$x_{\mathbb{C}_1 l_3}$ corresponds to the tangency point between the circle $\mathbb{C}_1$ and the line l_3. The equation (10) is used to determine the second tangent circle $\mathbb{C}_1$ in relation to the line l_3 :

$$\left\{x-\left[x_{\mathbb{C}_1 l_3}+\frac{\mathbb{R}_1 f'\left(x_{\mathbb{C}_1 l_3}\right)}{\sqrt{1+\left[f'\left(x_{\mathbb{C}_1 l_3}\right)\right]^2}}\right]\right\}^2+\left\{y-\left[f\left(x_{\mathbb{C}_1 l_3}\right)-\frac{\mathbb{R}_1}{\sqrt{1+\left[f'\left(x_{\mathbb{C}_1 l_3}\right)\right]^2}}\right]\right\}^2=\mathbb{R}_1^2 \quad (25)$$

The equations (15) and (16) are used to calculate $f\left(x_{\mathbb{C}_1 l_3}\right)$ and $f'\left(x_{\mathbb{C}_1 l_3}\right)$:

$$f\left(x_{\mathbb{C}_1 l_3}\right) = -5x_{\mathbb{C}_1 l_3} + 18 \tag{26}$$
$$f'\left(x_{\mathbb{C}_1 l_3}\right) = -5 \tag{27}$$

Replacing the equations (26) and (27) in the equation (25) :

$$\left\{x - \left[x_{\mathbb{C}_1 l_3} - \frac{5\mathbb{R}_1}{\sqrt{26}}\right]\right\}^2 + \left\{y - \left[-5x_{\mathbb{C}_1 l_3} + 18 - \frac{\mathbb{R}_1}{\sqrt{26}}\right]\right\}^2 = \mathbb{R}_1^2 \tag{28}$$

The equations (20),(24) and (28) are the equations of the tangent circle $\mathbb{C}_1$. The equations are the same. From the equations (20) and (24) :

$$x_{\mathbb{C}_1 l_1} + \frac{\mathbb{R}_1}{\sqrt{2}} = x_{\mathbb{C}_1 l_2} + \frac{\mathbb{R}_1}{\sqrt{2}} \tag{29}$$

From the equations (24) and (28) :

$$x_{\mathbb{C}_1 l_2} + \frac{\mathbb{R}_1}{\sqrt{2}} = x_{\mathbb{C}_1 l_3} - 5\frac{\mathbb{R}_1}{\sqrt{26}} \tag{30}$$

From the equations (20) and (24) :

$$x_{\mathbb{C}_1 l_1} + 3 - \frac{\mathbb{R}_1}{\sqrt{2}} = -x_{\mathbb{C}_1 l_2} + 1 + \frac{\mathbb{R}_1}{\sqrt{2}} \tag{31}$$

From the equations (24) and (28) :

$$-x_{\mathbb{C}_1 l_2} + 1 + \frac{\mathbb{R}_1}{\sqrt{2}} = -5x_{\mathbb{C}_1 l_3} + 18 - \frac{\mathbb{R}_1}{\sqrt{26}} \tag{32}$$

There are four equations ((29), (30), (31) and (32)) with four unknowns $\mathbb{R}_1, x_{\mathbb{C}_1 l_1}, x_{\mathbb{C}_1 l_2}$ and $x_{\mathbb{C}_1 l_3}$. Solving the system of equations (equations (29)-(32)) (Appendix B.1):

$$x_{\mathbb{C}_1 l_1} = \frac{21}{10 + 2\sqrt{13}} - 1 \tag{33}$$
$$x_{\mathbb{C}_1 l_2} = \frac{21}{10 + 2\sqrt{13}} - 1 \tag{34}$$
$$x_{\mathbb{C}_1 l_3} = \frac{21}{5 + \sqrt{13}} + \frac{105}{10\sqrt{13} + 26} - 1 \tag{35}$$
$$\mathbb{R}_1 = \frac{21}{5\sqrt{2} + \sqrt{26}} \tag{36}$$
$$h_{\mathbb{C}_1} = \frac{21}{5 + \sqrt{13}} - 1 \tag{37}$$
$$k_{\mathbb{C}_1} = 2 \tag{38}$$

In the figure 1b is plotted the tangent circle $\mathbb{C}_1$.

Other tangent circles can be determined. Using a similar procedure the tangent circle $\mathbb{C}_2$ is calculated. $x_{\mathbb{C}_2 l_1}$ corresponds to the tangency point between the circle $\mathbb{C}_2$ and the line l_1. The equation (10) is used to determine the second tangent circle $\mathbb{C}_2$ in relation to the line l_1 :

$$\left\{x-\left[x_{\mathbb{C}_2 l_1}+\frac{\mathbb{R}_2 f'\left(x_{\mathbb{C}_2 l_1}\right)}{\sqrt{1+\left[f'\left(x_{\mathbb{C}_2 l_1}\right)\right]^2}}\right]\right\}^2+\left\{y-\left[f\left(x_{\mathbb{C}_2 l_1}\right)-\frac{\mathbb{R}_2}{\sqrt{1+\left[f'\left(x_{\mathbb{C}_2 l_1}\right)\right]^2}}\right]\right\}^2=\mathbb{R}_2^2 \tag{39}$$

The equations (11) and (12) are used to calculate $f\left(x_{\mathbb{C}_2 l_1}\right)$ and $f'\left(x_{\mathbb{C}_2 l_1}\right)$:

$$f\left(x_{\mathbb{C}_2 l_1}\right)=x_{\mathbb{C}_2 l_1}+3 \tag{40}$$

$$f'\left(x_{\mathbb{C}_2 l_1}\right)=1 \tag{41}$$

Replacing the equations (40) and (41) in the equation (39):

$$\left\{x-\left[x_{\mathbb{C}_2 l_1}+\frac{\mathbb{R}_2}{\sqrt{2}}\right]\right\}^2+\left\{y-\left[x_{\mathbb{C}_2 l_1}+3-\frac{\mathbb{R}_2}{\sqrt{2}}\right]\right\}^2=\mathbb{R}_2^2 \tag{42}$$

The tangent circle $\mathbb{C}_2$ can also be calculated in relation to the lines l_2 and $l_3 \cdot x_{\mathbb{C}_2 l_2}$ corresponds to the tangency point between the circle $\mathbb{C}_2$ and the line l_2. The equation (5) is used to determine the first tangent circle $\mathbb{C}_2$ in relation to the line l_2 :

$$\left\{x-\left[x_{\mathbb{C}_2 l_2}-\frac{\mathbb{R}_2 f'\left(x_{\mathbb{C}_2 l_2}\right)}{\sqrt{1+\left[f'\left(x_{\mathbb{C}_2 l_2}\right)\right]^2}}\right]\right\}^2+\left\{y-\left[f\left(x_{\mathbb{C}_2 l_2}\right)+\frac{\mathbb{R}_2}{\sqrt{1+\left[f'\left(x_{\mathbb{C}_2 l_2}\right)\right]^2}}\right]\right\}^2=\mathbb{R}_2^2 \tag{43}$$

The equations (13) and (14) are used to calculate $f\left(x_{\mathbb{C}_2 l_2}\right)$ and $f'\left(x_{\mathbb{C}_2 l_2}\right)$:

$$f\left(x_{\mathbb{C}_2 l_2}\right)=-x_{\mathbb{C}_2 l_2}+1 \tag{44}$$

$$f'\left(x_{\mathbb{C}_2 l_2}\right)=-1 \tag{45}$$

Replacing the equations (44) and (45) in the equation (43) :

$$\left\{x-\left[x_{\mathbb{C}_2 l_2}+\frac{\mathbb{R}_2}{\sqrt{2}}\right]\right\}^2+\left\{y-\left[-x_{\mathbb{C}_2 l_2}+1+\frac{\mathbb{R}_2}{\sqrt{2}}\right]\right\}^2=\mathbb{R}_2^2 \tag{46}$$

$x_{\mathbb{C}_2 l_3}$ corresponds to the tangency point between the circle $\mathbb{C}_2$ and the line l_3. The equation (5) is used to determine the first tangent circle $\mathbb{C}_2$ in relation to the line l_3 :

$$\left\{x-\left[x_{\mathbb{C}_2 l_3}-\frac{\mathbb{R}_2 f'\left(x_{\mathbb{C}_2 l_3}\right)}{\sqrt{1+\left[f'\left(x_{\mathbb{C}_2 l_3}\right)\right]^2}}\right]\right\}^2+\left\{y-\left[f\left(x_{\mathbb{C}_2 l_3}\right)+\frac{\mathbb{R}_2}{\sqrt{1+\left[f'\left(x_{\mathbb{C}_2 l_3}\right)\right]^2}}\right]\right\}^2=\mathbb{R}_2^2 \tag{47}$$

The equations (15) and (16) are used to calculate $f\left(x_{\mathbb{C}_2 l_3}\right)$ and $f'\left(x_{\mathbb{C}_2 l_3}\right)$:

$$f\left(x_{\mathbb{C}_2 l_3}\right) = -5x_{\mathbb{C}_2 l_3} + 18 \tag{48}$$

$$f'\left(x_{\mathbb{C}_2 l_3}\right) = -5 \tag{49}$$

Replacing the equations (48) and (49) in the equation (47) :

$$\left\{x - \left[x_{\mathbb{C}_2 l_3} + 5\frac{\mathbb{R}_2}{\sqrt{26}}\right]\right\}^2 + \left\{y - \left[-5x_{\mathbb{C}_2 l_3} + 18 + \frac{\mathbb{R}_2}{\sqrt{26}}\right]\right\}^2 = \mathbb{R}_2^2 \tag{50}$$

The equations (42), (46) and (50) are the equations of the tangent circle $\mathbb{C}_2$. The equations are the same. From the equations (42) and (46) :

$$x_{\mathbb{C}_2 l_1} + \frac{\mathbb{R}_2}{\sqrt{2}} = x_{\mathbb{C}_2 l_2} + \frac{\mathbb{R}_2}{\sqrt{2}} \tag{51}$$

From the equations (46) and (50):

$$x_{\mathbb{C}_2 l_2} + \frac{\mathbb{R}_2}{\sqrt{2}} = x_{\mathbb{C}_2 l_3} + 5\frac{\mathbb{R}_2}{\sqrt{26}} \tag{52}$$

From the equations (42) and (46) :

$$x_{\mathbb{C}_2 l_1} + 3 - \frac{\mathbb{R}_2}{\sqrt{2}} = -x_{\mathbb{C}_2 l_2} + 1 + \frac{\mathbb{R}_2}{\sqrt{2}} \tag{53}$$

From the equations (46) and (50) :

$$-x_{\mathbb{C}_2 l_2} + 1 + \frac{\mathbb{R}_2}{\sqrt{2}} = -5x_{\mathbb{C}_2 l_3} + 18 + \frac{\mathbb{R}_2}{\sqrt{26}} \tag{54}$$

There are four equations ((51),(52),(53) and (54)) with four unknowns $\mathbb{R}_2, x_{\mathbb{C}_2 l_1}, x_{\mathbb{C}_2 l_2}$ and $x_{\mathbb{C}_2 l_3}$. Solving the system of equations (equations (51)-(54)) (Appendix B.2):

$$x_{\mathbb{C}_2 l_1} = \frac{21}{10 - 2\sqrt{13}} - 1 \tag{55}$$

$$x_{\mathbb{C}_2 l_2} = \frac{21}{10 - 2\sqrt{13}} - 1 \tag{56}$$

$$x_{\mathbb{C}_2 l_3} = \frac{21}{5 - \sqrt{13}} - \frac{105}{10\sqrt{13} - 26} - 1 \tag{57}$$

$$\mathbb{R}_2 = \frac{21}{5\sqrt{2} - \sqrt{26}} \tag{58}$$

$$h_{\mathbb{C}_2} = \frac{21}{5 - \sqrt{13}} - 1 \tag{59}$$

$$k_{\mathbb{C}_2} = 2 \tag{60}$$

In the figure 1c is plotted the tangent circle $\mathbb{C}_2$. The tangent circle $\mathbb{C}_3$ is calculated. $x_{\mathbb{C}_3 l_1}$ corresponds to the tangency point between the circle $\mathbb{C}_3$ and the line l_1. The equation (5) is used to determine the first tangent circle $\mathbb{C}_3$ in relation to the line l_1 :

$$\left\{x - \left[x_{\mathbb{C}_3 l_1} - \frac{\mathbb{R}_3 f'\left(x_{\mathbb{C}_3 l_1}\right)}{\sqrt{1+\left[f'\left(x_{\mathbb{C}_3 l_1}\right)\right]^2}}\right]\right\}^2 + \left\{y - \left[f\left(x_{\mathbb{C}_3 l_1}\right) + \frac{\mathbb{R}_3}{\sqrt{1+\left[f'\left(x_{\mathbb{C}_3 l_1}\right)\right]^2}}\right]\right\}^2 = \mathbb{R}_3^2 \tag{61}$$

The equations (11) and (12) are used to calculate $f\left(x_{\mathbb{C}_3 l_1}\right)$ and $f'\left(x_{\mathbb{C}_3 l_1}\right)$:

$$f\left(x_{\mathbb{C}_3 l_1}\right) = x_{\mathbb{C}_3 l_1} + 3 \tag{62}$$

$$f'\left(x_{\mathbb{C}_3 l_1}\right) = 1 \tag{63}$$

Replacing the equations (62) and (63) in the equation (61) :

$$\left\{x - \left[x_{\mathbb{C}_3 l_1} - \frac{\mathbb{R}_3}{\sqrt{2}}\right]\right\}^2 + \left\{y - \left[x_{\mathbb{C}_3 l_1} + 3 + \frac{\mathbb{R}_3}{\sqrt{2}}\right]\right\}^2 = \mathbb{R}_3^2 \tag{64}$$

The tangent circle $\mathbb{C}_3$ can also be calculated in relation to the lines l_2 and $l_3 \cdot x_{\mathbb{C}_3 l_2}$ corresponds to the tangency point between the circle $\mathbb{C}_3$ and the line l_2. The equation (5) is used to determine the first tangent circle $\mathbb{C}_3$ in relation to the line l_2 :

$$\left\{x - \left[x_{\mathbb{C}_3 l_2} - \frac{\mathbb{R}_3 f'\left(x_{\mathbb{C}_3 l_2}\right)}{\sqrt{1+\left[f'\left(x_{\mathbb{C}_3 l_2}\right)\right]^2}}\right]\right\}^2 + \left\{y - \left[f\left(x_{\mathbb{C}_3 l_2}\right) + \frac{\mathbb{R}_3}{\sqrt{1+\left[f'\left(x_{\mathbb{C}_3 l_2}\right)\right]^2}}\right]\right\}^2 = \mathbb{R}_3^2 \tag{65}$$

The equations (13) and (14) are used to calculate $f\left(x_{\mathbb{C}_3 l_2}\right)$ and $f'\left(x_{\mathbb{C}_3 l_2}\right)$:

$$f\left(x_{\mathbb{C}_3 l_2}\right) = -x_{\mathbb{C}_3 l_2} + 1 \tag{66}$$

$$f'\left(x_{\mathbb{C}_3 l_2}\right) = -1 \tag{67}$$

Replacing the equations (66) and (67) in the equation (65) :

$$\left\{x - \left[x_{\mathbb{C}_3 l_2} + \frac{\mathbb{R}_3}{\sqrt{2}}\right]\right\}^2 + \left\{y - \left[-x_{\mathbb{C}_3 l_2} + 1 + \frac{\mathbb{R}_3}{\sqrt{2}}\right]\right\}^2 = \mathbb{R}_3^2 \tag{68}$$

$x_{\mathbb{C}_3 l_3}$ corresponds to the tangency point between the circle $\mathbb{C}_3$ and the line l_3. The equation (10) is used to determine the second tangent circle $\mathbb{C}_3$ in relation to the line l_3 :

$$\left\{x - \left[x_{\mathbb{C}_3 l_3} + \frac{\mathbb{R}_3 f'\left(x_{\mathbb{C}_3 l_3}\right)}{\sqrt{1+\left[f'\left(x_{\mathbb{C}_3 l_3}\right)\right]^2}}\right]\right\}^2 + \left\{y - \left[f\left(x_{\mathbb{C}_3 l_3}\right) - \frac{\mathbb{R}_3}{\sqrt{1+\left[f'\left(x_{\mathbb{C}_3 l_3}\right)\right]^2}}\right]\right\}^2 = \mathbb{R}_3^2 \tag{69}$$

The equations (15) and (16) are used to calculate $f\left(x_{\mathbb{C}_3 l_3}\right)$ and $f'\left(x_{\mathbb{C}_3 l_3}\right)$:

$$f\left(x_{\mathbb{C}_3 l_3}\right) = -5x_{\mathbb{C}_3 l_3} + 18 \tag{70}$$

$$f'\left(x_{\mathbb{C}_3 l_3}\right) = -5 \tag{71}$$

Replacing the equations (70) and (71) in the equation (69) :

$$\left\{x - \left[x_{\mathbb{C}_3 l_3} - 5\frac{\mathbb{R}_3}{\sqrt{26}}\right]\right\}^2 + \left\{y - \left[-5x_{\mathbb{C}_3 l_3} + 18 - \frac{\mathbb{R}_3}{\sqrt{26}}\right]\right\}^2 = \mathbb{R}_3^2 \tag{72}$$

The equations (64), (65) and (72) are the equations of the tangent circle $\mathbb{C}_3$. The equations are the same. From the equations (64) and (68) :

$$x_{\mathbb{C}_3 l_1} - \frac{\mathbb{R}_3}{\sqrt{2}} = x_{\mathbb{C}_3 l_2} + \frac{\mathbb{R}_3}{\sqrt{2}} \tag{73}$$

From the equations (68) and (72) :

$$x_{\mathbb{C}_3 l_2} + \frac{\mathbb{R}_3}{\sqrt{2}} = x_{\mathbb{C}_3 l_3} - 5\frac{\mathbb{R}_3}{\sqrt{26}} \tag{74}$$

From the equations (64) and (68) :

$$x_{\mathbb{C}_3 l_1} - \frac{\mathbb{R}_3}{\sqrt{2}} = x_{\mathbb{C}_3 l_2} + \frac{\mathbb{R}_3}{\sqrt{2}} \tag{75}$$

From the equations (68) and (72) :

$$x_{\mathbb{C}_3 l_2} + \frac{\mathbb{R}_3}{\sqrt{2}} = x_{\mathbb{C}_3 l_3} - 5\frac{\mathbb{R}_3}{\sqrt{26}} \tag{76}$$

There are four equations ((73), (74), (75) and (76)) with four unknowns $\mathbb{R}_3, x_{\mathbb{C}_3 l_1}, x_{\mathbb{C}_3 l_2}$ and $x_{\mathbb{C}_3 l_3}$. Solving the system of equations (equations (73)-(76)) (Appendix B.3):

$$x_{\mathbb{C}_3 l_1} = -1 - \frac{21}{2\sqrt{13}+2} + \frac{21}{\sqrt{13}+1} \tag{77}$$

$$x_{\mathbb{C}_3 l_2} = -1 - \frac{21}{2\sqrt{13}+2} \tag{78}$$

$$x_{\mathbb{C}_3 l_3} = -1 + \frac{105}{26+2\sqrt{13}} \tag{79}$$

$$\mathbb{R}_3 = \frac{21}{\sqrt{26}+\sqrt{2}} \tag{80}$$

$$h_{\mathbb{C}_3} = -1 \tag{81}$$

$$k_{\mathbb{C}_3} = 2 + \frac{21}{\sqrt{13}+1} \tag{82}$$

In the figure 1d is plotted the tangent circle $\mathbb{C}_3$.

The tangent circle $\mathbb{C}_4$ is calculated. $x_{\mathbb{C}_4 l_1}$ corresponds to the tangency point between the circle $\mathbb{C}_4$ and the line l_1. The equation (10) is used to determine the second tangent circle $\mathbb{C}_4$ in relation to the line l_1 :

$$\left\{x - \left[x_{\mathbb{C}_4 l_1} + \frac{\mathbb{R}_4 f'\left(x_{\mathbb{C}_4 l_1}\right)}{\sqrt{1+\left[f'\left(x_{\mathbb{C}_4 l_1}\right)\right]^2}}\right]\right\}^2 + \left\{y - \left[f\left(x_{\mathbb{C}_4 l_1}\right) - \frac{\mathbb{R}_4}{\sqrt{1+\left[f'\left(x_{\mathbb{C}_4 l_1}\right)\right]^2}}\right]\right\}^2 = \mathbb{R}_4^2 \tag{83}$$

The equations (11) and (12) are used to calculate $f\left(x_{\mathbb{C}_4 l_1}\right)$ and $f'\left(x_{\mathbb{C}_4 l_1}\right)$:

$$f\left(x_{\mathbb{C}_4 l_1}\right) = x_{\mathbb{C}_4 l_1} + 3 \tag{84}$$

$$f'\left(x_{\mathbb{C}_4 l_1}\right) = 1 \tag{85}$$

Replacing the equations (84) and (85) in the equation (83):

$$\left\{x - \left[x_{\mathbb{C}_4 l_1} + \frac{\mathbb{R}_4}{\sqrt{2}}\right]\right\}^2 + \left\{y - \left[x_{\mathbb{C}_4 l_1} + 3 - \frac{\mathbb{R}_4}{\sqrt{2}}\right]\right\}^2 = \mathbb{R}_4^2 \tag{86}$$

The tangent circle $\mathbb{C}_4$ can also be calculated in relation to the lines l_2 and $l_3 \cdot x_{\mathbb{C}_4 l_2}$ corresponds to the tangency point between the circle $\mathbb{C}_4$ and the line l_2. The equation (10) is used to determine the second tangent circle $\mathbb{C}_4$ in relation to the line l_2 :

$$\left\{x - \left[x_{\mathbb{C}_4 l_2} + \frac{\mathbb{R}_4 f'\left(x_{\mathbb{C}_4 l_2}\right)}{\sqrt{1+\left[f'\left(x_{\mathbb{C}_4 l_2}\right)\right]^2}}\right]\right\}^2 + \left\{y - \left[f\left(x_{\mathbb{C}_4 l_2}\right) - \frac{\mathbb{R}_4}{\sqrt{1+\left[f'\left(x_{\mathbb{C}_4 l_2}\right)\right]^2}}\right]\right\}^2 = \mathbb{R}_4^2 \tag{87}$$

The equations (13) and (14) are used to calculate $f\left(x_{\mathbb{C}_4 l_2}\right)$ and $f'\left(x_{\mathbb{C}_4 l_2}\right)$:

$$f\left(x_{\mathbb{C}_4 l_2}\right) = -x_{\mathbb{C}_4 l_2} + 1 \tag{88}$$

$$f'\left(x_{\mathbb{C}_4 l_2}\right) = -1 \tag{89}$$

Replacing the equations (88) and (89) in the equation (87) :

$$\left\{x - \left[x_{\mathbb{C}_4 l_2} - \frac{\mathbb{R}_4}{\sqrt{2}}\right]\right\}^2 + \left\{y - \left[-x_{\mathbb{C}_4 l_2} + 1 - \frac{\mathbb{R}_4}{\sqrt{2}}\right]\right\}^2 = \mathbb{R}_4^2 \tag{90}$$

$x_{\mathbb{C}_4 l_3}$ corresponds to the tangency point between the circle $\mathbb{C}_4$ and the line l_3. The equation (10) is used to determine the second tangent circle $\mathbb{C}_4$ in relation to the line l_3 :

$$\left\{x - \left[x_{\mathbb{C}_4 l_3} + \frac{\mathbb{R}_4 f'\left(x_{\mathbb{C}_4 l_3}\right)}{\sqrt{1+\left[f'\left(x_{\mathbb{C}_4 l_3}\right)\right]^2}}\right]\right\}^2 + \left\{y - \left[f\left(x_{\mathbb{C}_4 l_3}\right) - \frac{\mathbb{R}_4}{\sqrt{1+\left[f'\left(x_{\mathbb{C}_4 l_3}\right)\right]^2}}\right]\right\}^2 = \mathbb{R}_4^2 \tag{91}$$

The equations (15) and (16) are used to calculate $f\left(x_{\mathbb{C}_4 l_3}\right)$ and $f'\left(x_{\mathbb{C}_4 l_3}\right)$:

$$f\left(x_{\mathbb{C}_4 l_3}\right) = -5x_{\mathbb{C}_4 l_3} + 18 \tag{92}$$

$$f'\left(x_{\mathbb{C}_4 l_3}\right) = -5 \tag{93}$$

Replacing the equations (92) and (93) in the equation (91) :

$$\left\{x - \left[x_{\mathbb{C}_4 l_3} - 5\frac{\mathbb{R}_4}{\sqrt{26}}\right]\right\}^2 + \left\{y - \left[-5x_{\mathbb{C}_4 l_3} + 18 - \frac{\mathbb{R}_4}{\sqrt{26}}\right]\right\}^2 = \mathbb{R}_4^2 \tag{94}$$

The equations (86), (90) and (94) are the equations of the tangent circle $\mathbb{C}_4$. The equations are the same. From the equations (86) and (90) :

$$x_{\mathbb{C}_4 l_1} + \frac{\mathbb{R}_4}{\sqrt{2}} = x_{\mathbb{C}_4 l_2} - \frac{\mathbb{R}_4}{\sqrt{2}} \tag{95}$$

From the equations (90) and (94) :

$$x_{\mathbb{C}_4 l_2} - \frac{\mathbb{R}_4}{\sqrt{2}} = x_{\mathbb{C}_4 l_3} - 5\frac{\mathbb{R}_4}{\sqrt{26}} \tag{96}$$

From the equations (86) and (90) :

$$x_{\mathbb{C}_4 l_1} + 3 - \frac{\mathbb{R}_4}{\sqrt{2}} = -x_{\mathbb{C}_4 l_2} + 1 - \frac{\mathbb{R}_4}{\sqrt{2}} \tag{97}$$

From the equations (90) and (94) :

$$-x_{\mathbb{C}_4 l_2} + 1 - \frac{\mathbb{R}_4}{\sqrt{2}} = -5x_{\mathbb{C}_4 l_3} + 18 - \frac{\mathbb{R}_4}{\sqrt{26}} \tag{98}$$

There are four equations ((95), (96), (97) and (98)) with four unknowns $\mathbb{R}_4, x_{\mathbb{C}_4 l_1}, x_{\mathbb{C}_4 l_2}$ and $x_{\mathbb{C}_4 l_3}$. Solving the system of equations (equations (95)-(98)) (Appendix B.4):

$$x_{\mathbb{C}_4 l_1} = -1 + \frac{21}{2\sqrt{13}-2} - \frac{21}{\sqrt{13}-1} \tag{99}$$

$$x_{\mathbb{C}_4 l_2} = -1 + \frac{21}{2\sqrt{13}-2} \tag{100}$$

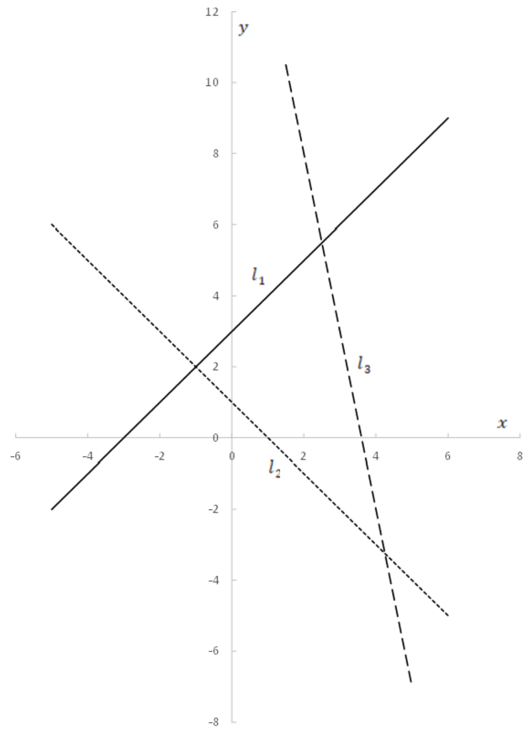
$$x_{\mathbb{C}_4 l_3} = -1 + \frac{105}{26-2\sqrt{13}} \tag{101}$$

$$\mathbb{R}_4 = \frac{21}{\sqrt{26}-\sqrt{2}} \tag{102}$$

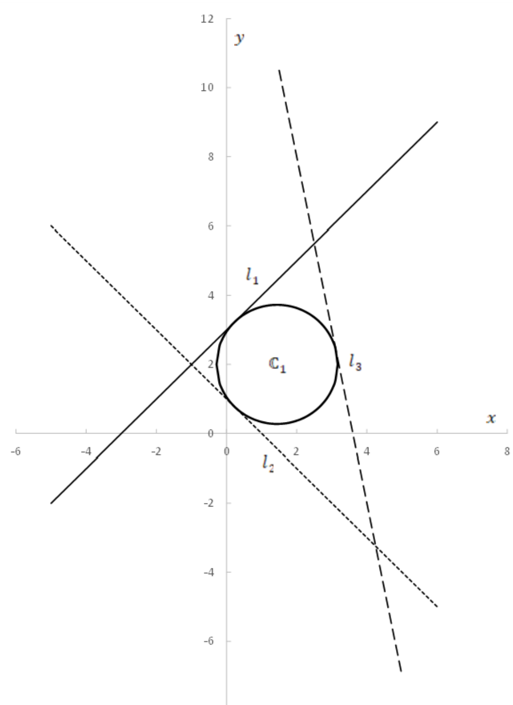
$$h_{\mathbb{C}_4} = -1 \tag{103}$$

$$k_{\mathbb{C}_4} = 2 - \frac{21}{\sqrt{13}-1} \tag{104}$$

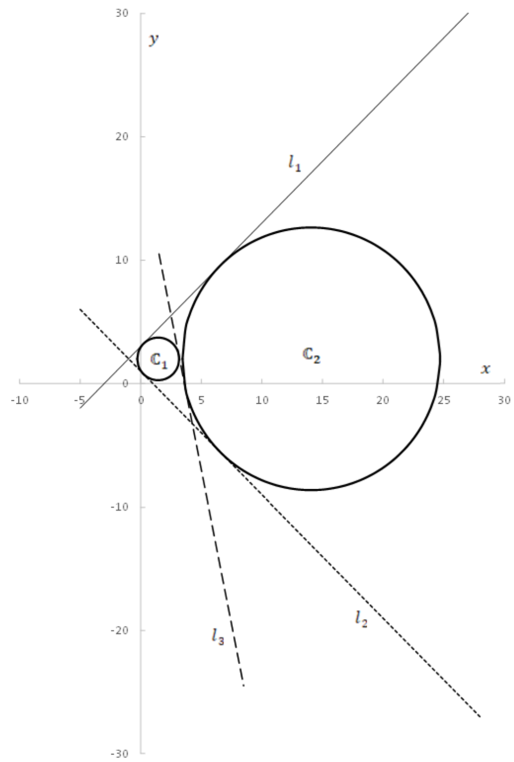
In the figure 1e is plotted the tangent circle $\mathbb{C}_4$. The derivatives at the tangency points for the tangent circle $\mathbb{C}_1$ can be found in the Appendix C.



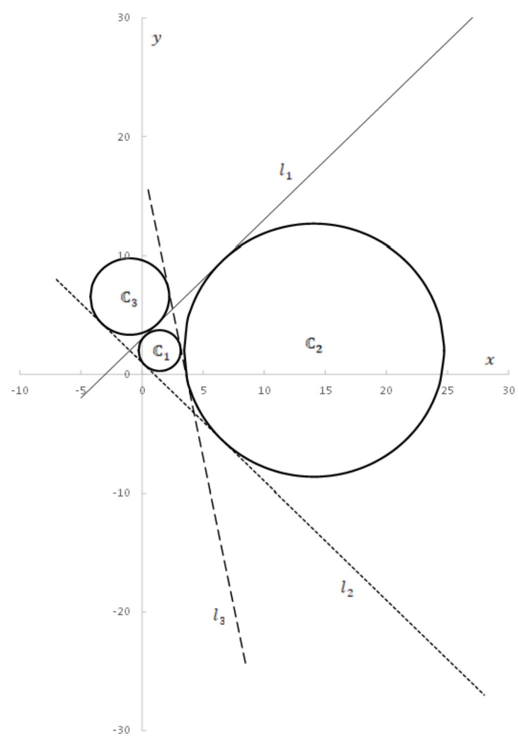
(a) The lines l_1, l_2 and l_3.



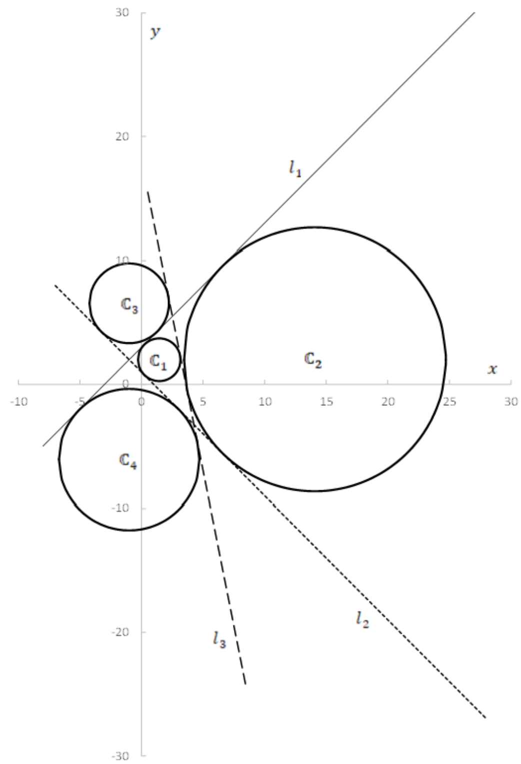
(b) The tangent circle $\mathbb{C}_1$.



(c) The tangent circle $\mathbb{C}_2$.



(d) The tangent circle $\mathbb{C}_3$.



(e) The tangent circle $\mathbb{C}_4$.

Fig. 1: Three lines (LLL).
Source: Own creation

3.2 One line and two points (LPP)

The circles through two points P_1, P_2 and tangent to a given line l are found in this case. The points $P_1(1,7), P_2(6,8)$ and a given line $l, y = x - 5$, are shown in the figure 2a.

The equation of the line is taken as a function and its derivative is calculated:

$$f(x) = y = x - 5(\text{ line } l\ ) \qquad (105)$$
$$f'(x) = y' = 1 \qquad (106)$$

A second tangent circle does not originate a possible solution. The tangent circle $\mathbb{C}_1$ is calculated. $x_{\mathbb{C}_1 l}$ corresponds to the tangency point between the tangent circle $\mathbb{C}_1$ and the given line l. The equation (5) is used to determine the first tangent circle $\mathbb{C}_1$ in relation to the given line l :

$$\left\{x-\left[x_{\mathbb{C}_1 l}-\frac{\mathbb{R}_1 f'\left(x_{\mathbb{C}_1 l}\right)}{\sqrt{1+\left[f'\left(x_{\mathbb{C}_1 l}\right)\right]^2}}\right]\right\}^2+\left\{y-\left[f\left(x_{\mathbb{C}_1 l}\right)+\frac{\mathbb{R}_1}{\sqrt{1+\left[f'\left(x_{\mathbb{C}_1 l}\right)\right]^2}}\right]\right\}^2=\mathbb{R}_1^2 \tag{107}$$

The equations (105) and (106) are used to calculate $f\left(x_{\mathbb{C}_1 l}\right)$ and $f'\left(x_{\mathbb{C}_1 l}\right)$:

$$f\left(x_{\mathbb{C}_1 l}\right)=x_{\mathbb{C}_1 l}-5 \tag{108}$$

$$f'\left(x_{\mathbb{C}_1 l}\right)=1 \tag{109}$$

Replacing the equations (108) and (109) in the equation (107) :

$$\left\{x-\left[x_{\mathbb{C}_1 l}-\frac{\mathbb{R}_1}{\sqrt{2}}\right]\right\}^2+\left\{y-\left[\left(x_{\mathbb{C}_1 l}-5\right)+\frac{\mathbb{R}_1}{\sqrt{2}}\right]\right\}^2=\mathbb{R}_1^2 \tag{110}$$

The circle (equation (110)) contains the point $P_1(1,7)$:

$$\left\{1-\left[x_{\mathbb{C}_1 l}-\frac{\mathbb{R}_1}{\sqrt{2}}\right]\right\}^2+\left\{7-\left[\left(x_{\mathbb{C}_1 l}-5\right)+\frac{\mathbb{R}_1}{\sqrt{2}}\right]\right\}^2=\mathbb{R}_1^2 \tag{111}$$

The circle (equation (110)) contains also the point $P_2(6,8)$:

$$\left\{6-\left[x_{\mathbb{C}_1 l}-\frac{\mathbb{R}_1}{\sqrt{2}}\right]\right\}^2+\left\{8-\left[\left(x_{\mathbb{C}_1 l}-5\right)+\frac{\mathbb{R}_1}{\sqrt{2}}\right]\right\}^2=\mathbb{R}_1^2 \tag{112}$$

There are two equations ((111) and (112)) with two unknowns $\mathbb{R}_1$ and $x_{\mathbb{C}_1 l}$. Solving the system of equations (equations (111)-(112)) (Appendix D). There are two tangent circles as solutions for $\mathbb{C}_1\left(\mathbb{C}_{1+}\right.$ and $\left.\mathbb{C}_{1-}\right)$. $\mathbb{C}_{1+}$ with radius $\mathbb{R}_{1+}$ and center $\left(h_{\mathbb{C}_1+}, k_{\mathbb{C}_1+}\right)$ and $\mathbb{C}_{1-}$ with radius $\mathbb{R}_{1-}$ and center $\left(h_{\mathbb{C}_1-}, k_{\mathbb{C}_1-}\right)$:

$$x_{\mathbb{C}_1 l+} = \frac{59 + \sqrt{1001}}{4} \tag{113}$$

$$\mathbb{R}_{1+} = \frac{\sqrt{2}\left[3\left(\frac{59+\sqrt{1001}}{4}\right) - 15\right]}{2} \tag{114}$$

$$h_{\mathbb{C}_1+} = \frac{1 - \sqrt{1001}}{8} \tag{115}$$

$$k_{\mathbb{C}_1+} = \frac{195 + 5\sqrt{1001}}{8} \tag{116}$$

$$x_{\mathbb{C}_1 l-} = \frac{59 - \sqrt{1001}}{4} \tag{117}$$

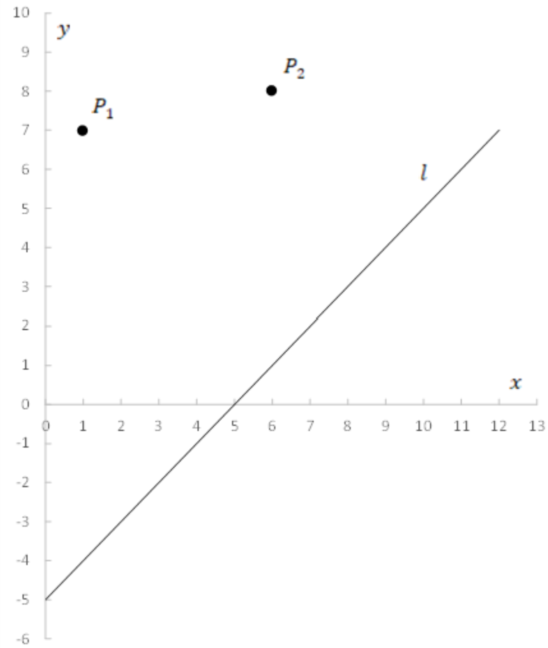
$$\mathbb{R}_{1-} = \frac{\sqrt{2}\left[3\left(\frac{59-\sqrt{1001}}{4}\right) - 15\right]}{2} \tag{118}$$

$$h_{\mathbb{C}_1-} = \frac{1 + \sqrt{1001}}{8} \tag{119}$$

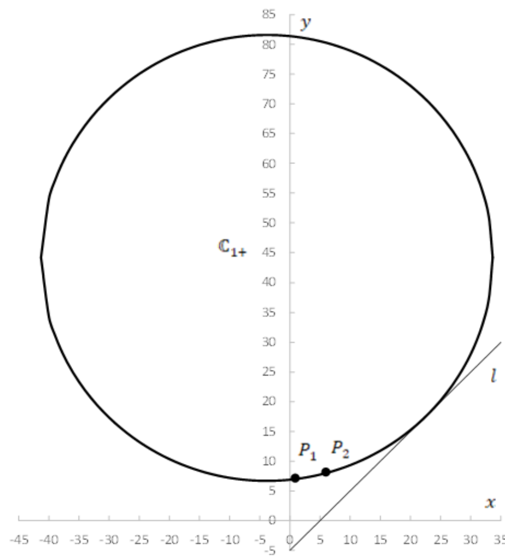
$$k_{\mathbb{C}_1-} = \frac{195 - 5\sqrt{1001}}{8} \tag{120}$$

In the figures 2b, 2c and 2d are plotted the tangent circles $\mathbb{C}_{1+}$ and $\mathbb{C}_{1-}$. The derivatives at the tangency points for the tangent circle $\mathbb{C}_{1+}$ can be found in the Appendix E.

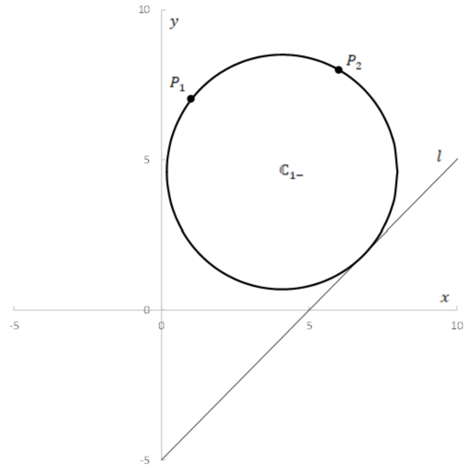
In the figures 2b, 2c and 2d are plotted the tangent circles $\mathbb{C}_{1+}$ and $\mathbb{C}_{1-}$. The derivatives at the tangency points for the tangent circle $\mathbb{C}_{1+}$ can be found in the Appendix E.



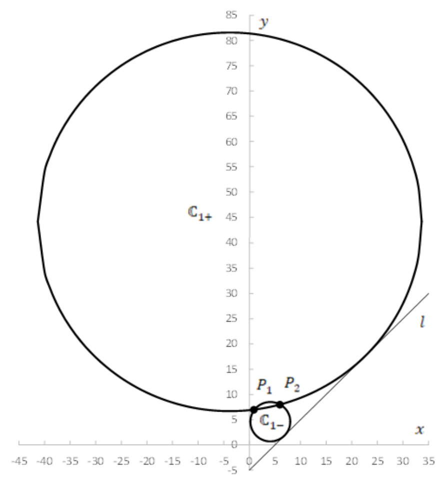
(a) Two points P_1, P_2 and the given line l.



(b) The tangent circle $\mathbb{C}_{1+}$.



(c) The tangent circle $\mathbb{C}_{1-}$.



(d) The tangent circle $\mathbb{C}_{1+}$ and $\mathbb{C}_{1-}$.

Fig. 2: One line and two points (LPP).
Source: Own creation

3.3 Three circles (CCC)

The circles tangent to three circles C_1, C_2 and C_3 are found in this case. The circles C_1, C_2 and C_3 are shown in the figure 3a. The equations of the circles are taken as functions and their derivatives are calculated:

$$f(x) = y = +\sqrt{4 - x^2} \quad \text{(upper semicircle } C_1\text{)} \tag{121}$$
$$f'(x) = y' = -\frac{x}{\sqrt{4 - x^2}} \tag{122}$$
$$f(x) = y = -\sqrt{4 - x^2} \quad \text{(lower semicircle } C_1\text{)} \tag{123}$$
$$f'(x) = y' = \frac{x}{\sqrt{4 - x^2}} \tag{124}$$
$$f(x) = y = +\sqrt{9 - (x - 7)^2} - 1 \quad \text{(upper semicircle } C_2\text{)} \tag{125}$$
$$f'(x) = y' = -\frac{x - 7}{\sqrt{9 - (x - 7)^2}} \tag{126}$$

$$f(x) = y = -\sqrt{9 - (x - 7)^2} - 1 \quad \text{(lower semicircle } C_2\text{)} \tag{127}$$
$$f'(x) = y' = \frac{x - 7}{\sqrt{9 - (x - 7)^2}} \tag{128}$$
$$f(x) = y = +\sqrt{16 - (x - 3)^2} + 10 \quad \text{(upper semicircle } C_3 \text{)} \tag{129}$$
$$f'(x) = y' = -\frac{x - 3}{\sqrt{16 - (x - 3)^2}} \tag{130}$$
$$f(x) = y = -\sqrt{16 - (x - 3)^2} + 10 \quad \text{(lower semicircle } C_3 \text{)} \tag{131}$$
$$f'(x) = y' = \frac{x - 3}{\sqrt{16 - (x - 3)^2}} \tag{132}$$

The possible combinations of the different functions of the semicircles C_1, C_2 and C_3 are shown in the figure 3b. The plus + represents the upper semicircle and the minus - represents the lower semicircle. There are 8 combinations. The combination corresponding to the upper semicircle of C_1, the upper semicircle of C_2 and the lower semicircle C_3 is implemented (figure 3c).

The tangent circle $\mathbb{C}_1$ is calculated. $x_{\mathbb{C}_1 C_1}$ corresponds to the tangency point between the circle $\mathbb{C}_1$ and the given circle C_1. The equation (5) is used to determine the first tangent circle $\mathbb{C}_1$ in relation to the given circle C_1 :

$$\left\{x - \left[x_{\mathbb{C}_1 c_1} - \frac{\mathbb{R}_1 f'\left(x_{\mathbb{C}_1 C_1}\right)}{\sqrt{1+\left[f'\left(x_{\mathbb{C}_1 C_1}\right)\right]^2}}\right]\right\}^2 + \left\{y - \left[f\left(x_{\mathbb{C}_1 C_1}\right) + \frac{\mathbb{R}_1}{\sqrt{1+\left[f'\left(x_{\mathbb{C}_1 C_1}\right)\right]^2}}\right]\right\}^2 = \mathbb{R}_1^2 \tag{133}$$

The equations (121) and (122) are used to calculate $f\left(x_{\mathbb{C}_1 C_1}\right)$ and $f'\left(x_{\mathbb{C}_1 C_1}\right)$:

$$f\left(x_{\mathbb{C}_1 C_1}\right) = \sqrt{4 - x^2_{\mathbb{C}_1 C_1}} \tag{134}$$
$$f'\left(x_{\mathbb{C}_1 C_1}\right) = -\frac{x_{\mathbb{C}_1 C_1}}{\sqrt{4 - x^2_{\mathbb{C}_1 C_1}}} \tag{135}$$

Replacing the equations (134) and (135) in the equation (133):

$$\left\{x-\left[x_{\mathbb{C}_1C_1}+\frac{\frac{\mathbb{R}_1 x_{\mathbb{C}_1C_1}}{\sqrt{4-x^2_{\mathbb{C}_1C_1}}}}{\sqrt{1+\frac{x^2_{\mathbb{C}_1C_1}}{4-x^2_{\mathbb{C}_1C_1}}}}\right]\right\}^2+\left\{y-\left[\sqrt{4-x^2_{\mathbb{C}_1C_1}}+\frac{\mathbb{R}_1}{\sqrt{1+\frac{x^2_{\mathbb{C}_1C_1}}{4-x^2_{\mathbb{C}_1C_1}}}}\right]\right\}^2=\mathbb{R}_1^2 \tag{136}$$

Operating with the result (Appendix F.1):

$$\left\{x-\left[x_{\mathbb{C}_1C_1}+\frac{\mathbb{R}_1 x_{\mathbb{C}_1}C_1}{2}\right]\right\}^2+\left\{y-\left[\sqrt{4-x^2_{\mathbb{C}_1C_1}}\left(1+\frac{\mathbb{R}_1}{2}\right)\right]\right\}^2=\mathbb{R}_1^2 \tag{137}$$

The tangent circle $\mathbb{C}_1$ can also be calculated in relation to the given circles C_2 and $C_3 \cdot x_{\mathbb{C}_1}C_2$ corresponds to the tangency point between the circle $\mathbb{C}_1$ and the given circle C_2. The equation (5) is used to determine the first tangent circle $\mathbb{C}_1$ in relation to the given circle C_2 :

$$\left\{x-\left[x_{\mathbb{C}_1c_2}-\frac{\mathbb{R}_1 f'\left(x_{\mathbb{C}_1c_2}\right)}{\sqrt{1+\left[f'\left(x_{\mathbb{C}_1c_2}\right)\right]^2}}\right]\right\}^2+\left\{y-\left[f\left(x_{\mathbb{C}_1c_2}\right)+\frac{\mathbb{R}_1}{\sqrt{1+\left[f'\left(x_{\mathbb{C}_1c_2}\right)\right]^2}}\right]\right\}^2=\mathbb{R}_1^2 \tag{138}$$

The equations (125) and (126) are used to calculate $f\left(x_{\mathbb{C}_1C_2}\right)$ and $f'\left(x_{\mathbb{C}_1C_2}\right)$:

$$f\left(x_{\mathbb{C}_1C_2}\right)=\sqrt{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2}-1 \tag{139}$$

$$f'\left(x_{\mathbb{C}_1C_2}\right)=-\frac{x_{\mathbb{C}_1C_2-7}}{\sqrt{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2}} \tag{140}$$

Replacing the equations (139) and (140) in the equation (138):

$$\left\{x-\left[x_{\mathbb{C}_1C_2}+\frac{\frac{\mathbb{R}_1\left(x_{\mathbb{C}_1C_2-7}\right)}{\sqrt{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2}}}{\sqrt{1+\frac{\left(x_{\mathbb{C}_1C_2}-7\right)^2}{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2}}}\right]\right\}^2+\left\{y-\left[\sqrt{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2}-1+\frac{\mathbb{R}_1}{\sqrt{1+\frac{\left(x_{\mathbb{C}_1C_2}-7\right)^2}{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2}}}\right]\right\}^2=\mathbb{R}_1^2 \tag{141}$$

Operating with the result (Appendix F.2):

$$\left\{x-\left[x_{\mathbb{C}_1C_2}+\frac{\mathbb{R}_1\left(x_{\mathbb{C}_1}C_2-7\right)}{3}\right]\right\}^2+\left\{y-\left[\sqrt{9-\left(x_{\mathbb{C}_1c_2}-7\right)^2}\left(1+\frac{\mathbb{R}_1}{3}\right)-1\right]\right\}^2=\mathbb{R}_1^2 \tag{142}$$

$x_{\mathbb{C}_1C_3}$ corresponds to the tangency point between the circle $\mathbb{C}_1$ and the given circle C_3. The equation (10) is used to determine the second tangent circle $\mathbb{C}_1$ in relation to the given circle C_3 :

$$\left\{x-\left[x_{\mathbb{C}_1C_3}+\frac{\mathbb{R}_1 f'\left(x_{\mathbb{C}_1c_3}\right)}{\sqrt{1+\left[f'\left(x_{\mathbb{C}_1c_3}\right)\right]^2}}\right]\right\}^2+\left\{y-\left[f\left(x_{\mathbb{C}_1c_3}\right)-\frac{\mathbb{R}_1}{\sqrt{1+\left[f'\left(x_{\mathbb{C}_1c_3}\right)\right]^2}}\right]\right\}^2=\mathbb{R}_1^2 \tag{143}$$

The equations (131) and (132) are used to calculate $f\left(x_{\mathbb{C}_1 c_3}\right)$ and $f'\left(x_{\mathbb{C}_1 c_3}\right)$:

$$f\left(x_{\mathbb{C}_1 C_3}\right) = -\sqrt{16 - \left(x_{\mathbb{C}_1 C_3} - 3\right)^2} + 10 \tag{144}$$

$$f'\left(x_{\mathbb{C}_1 C_3}\right) = \frac{x_{\mathbb{C}_1 C_3 - 3}}{\sqrt{16 - \left(x_{\mathbb{C}_1 C_3} - 3\right)^2}} \tag{145}$$

Replacing the equations (144) and (145) in the equation (143):

$$\left\{x - \left[x_{\mathbb{C}_1 c_3} + \frac{\frac{\mathbb{R}_1\left(x_{\mathbb{C}_1 C_3} - 3\right)}{\sqrt{16 - \left(x_{\mathbb{C}_1 C_3} - 3\right)^2}}}{\sqrt{1 + \frac{\left(x_{\mathbb{C}_1 C_3} - 3\right)^2}{16 - \left(x_{\mathbb{C}_1 C_3} - 3\right)^2}}}\right]\right\}^2 + \left\{y - \left[-\sqrt{16 - \left(x_{\mathbb{C}_1 C_3} - 3\right)^2} + 10 - \frac{\mathbb{R}_1}{\sqrt{1 + \frac{\left(x_{\mathbb{C}_1 C_3} - 3\right)^2}{16 - \left(x_{\mathbb{C}_1 C_3} - 3\right)^2}}}\right]\right\}^2 = \mathbb{R}_1^2 \tag{146}$$

Operating with the result (Appendix F.3):

$$\left\{x - \left[x_{\mathbb{C}_1 c_3} + \frac{\mathbb{R}_1\left(x_{\mathbb{C}_1 c_3 - 3}\right)}{4}\right]\right\}^2 + \left\{y - \left[-\sqrt{16 - \left(x_{\mathbb{C}_1 c_3} - 3\right)^2}\left(1 + \tfrac{\mathbb{R}_1}{4}\right) + 10\right]\right\}^2 = \mathbb{R}_1^2 \tag{147}$$

The equations (137), (142) and (147) are the equations of the tangent circle $\mathbb{C}_1$. The equations are the same. From the equations (137) and (142):

$$x_{\mathbb{C}_1 C_1} + \frac{\mathbb{R}_1 x_{\mathbb{C}_1 C_1}}{2} = x_{\mathbb{C}_1 C_2} + \frac{\mathbb{R}_1\left(x_{\mathbb{C}_1 C_2 - 7}\right)}{3} \tag{148}$$

From the equations (142) and (147):

$$x_{\mathbb{C}_1 C_2} + \frac{\mathbb{R}_1\left(x_{\mathbb{C}_1 C_2} - 7\right)}{3} = x_{\mathbb{C}_1 C_3} + \frac{\mathbb{R}_1\left(x_{\mathbb{C}_1 C_3} - 3\right)}{4} \tag{149}$$

From the equations (137) and (142):

$$\sqrt{4 - x_{\mathbb{C}_1 C_1}^2}\left(1 + \frac{\mathbb{R}_1}{2}\right) = \sqrt{9 - \left(x_{\mathbb{C}_1 C_2} - 7\right)^2}\left(1 + \frac{\mathbb{R}_1}{3}\right) - 1 \tag{150}$$

From the equations (142) and (147):

$$\sqrt{9 - \left(x_{\mathbb{C}_1 C_2} - 7\right)^2}\left(1 + \tfrac{\mathbb{R}_1}{3}\right) - 1 = -\sqrt{16 - \left(x_{\mathbb{C}_1 C_3} - 3\right)^2}\left(1 + \tfrac{\mathbb{R}_1}{4}\right) + 10 \tag{151}$$

There are four equations ((148), (149), (150) and (151)) with four unknowns $\mathbb{R}_1, x_{\mathbb{C}_1 c_1}, x_{\mathbb{C}_1 c_2}$ and $x_{\mathbb{C}_1 C_3}$. Solving the system of equations (equations (148)-(151)) (Appendix G):

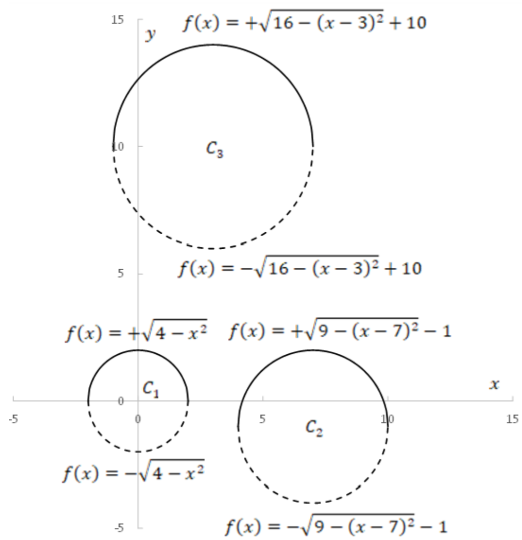
$$x_{\mathbb{C}_1 C_1} \approx 1,41074 \tag{152}$$
$$x_{\mathbb{C}_1 C_2} \approx 5,05040 \tag{153}$$
$$x_{\mathbb{C}_1 C_3} \approx 3,18254 \tag{154}$$
$$\mathbb{R}_1 \approx 2,68562 \tag{155}$$
$$h_{\mathbb{C}_1} \approx 3,30510 \tag{156}$$
$$k_{\mathbb{C}_1} \approx 3,32134 \tag{157}$$

In the figures 3d and 3e are plotted the tangent circle $\mathbb{C}_1$.

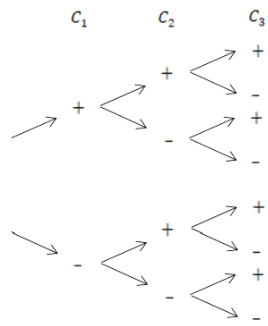
Other tangent circles to the three given circles C_1, C_2 and C_3 can be determined similarly using other combinations of the figure 3b.

The derivatives at the tangency points for the tangent circle $\mathbb{C}_1$ can be found in the Appendix H.

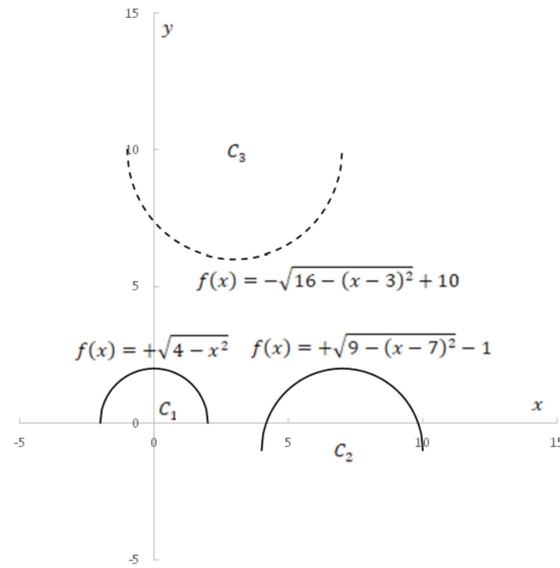
The results here obtained for the case 3.3 Three circles (CCC) are compared with the results applying the method in (Courant & Robbins, 1941, p. 125-127). The values of the radius $\mathbb{R}_1$ and the center $(h_{\mathbb{C}_1}, k_{\mathbb{C}_1})$ are the same using both methods for the tangent circle $\mathbb{C}_1 \cdot \mathbb{R}_1$ can be obtained as the roots of a quadratic equation using the method in (Courant & Robbins, 1941, p. 125-127) and $\mathbb{R}_1$ can be obtained as the roots of a quartic equation using this method. Interestingly, the roots of the quartic equation contain the roots of the quadratic equation.



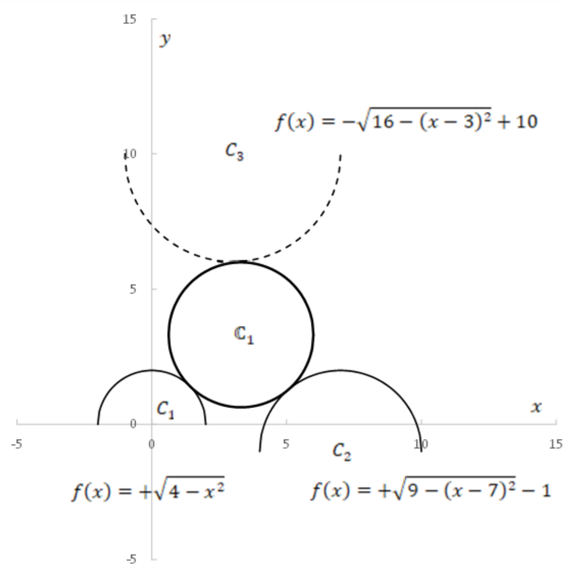
(a) The circles C_1, C_2 and C_3.



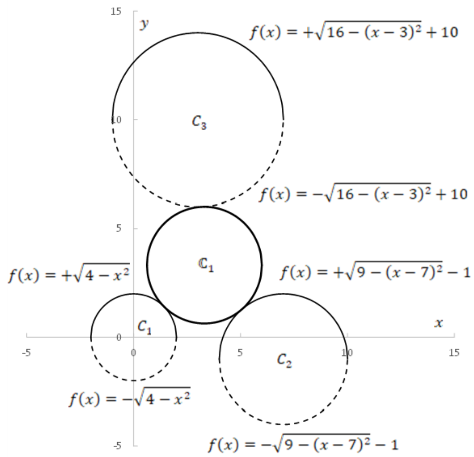
(b) Possible combinations of the different functions of the semicircles C_1, C_2 and C_3.



(c) Combination corresponding to the upper semicircle of C_1, the upper semicircle of C_2 and the lower semicircle C_3.



(d) The tangent circle $\mathbb{C}_1$.



(e) The tangent circle $\mathbb{C}_1$.

Fig. 3: Three circles (CCC).
Source: Own creation

4 Conclusions

A new method is presented to solve the Apollonius' problem. Tangent circles to the graph of a function can be used to solve this problem. Three types of the problem, three lines (LLL), one line and two points (LPP) and three circles (CCC) are solved. The solution is similar in the other seven cases of the problem.

When the objects, line or circle, are part of the elements of the problem, the line or circle are taken as functions. When a point is an element of the problem, the equation of the tangent circle must contain this point. The equations of the tangent circles in the form center-radius are applied to these functions. At this point, the tangent circles to the functions are determined. Since the unknown tangent circle is tangent to the other objects (or passes through the eventual given points) of the problem, comparing the different equations produces a system of non-linear equations. The solution of this system of equations gives the radius of the unknown tangent circle and the points of tangency. The center of the tangent circle can be obtained from the previous results and any of the equations in the form center-radius.

The values of the derivatives at the points of tangency for the given elements (lines or circles) and the obtained tangent circle are the same. The results here obtained (center-radius) for the case three circles (CCC) and the results from other previously published method are the same.

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Appendix A. Equations of tangent circles in the form center-radius

A1. Addition of the square of the expressions for the first tangent circle with radius r to the graph of a function f at the point $P(x_0, f(x_0))$ (Gómez-Villarraga, F. 2021)

$$\left\{x(t)-\left[x_0-\frac{rf'(x_0)}{\sqrt{1+[f'(x_0)]^2}}\right]\right\}^2+\left\{y(t)-\left[f(x_0)+\frac{r}{\sqrt{1+[f'(x_0)]^2}}\right]\right\}^2=$$

$$\left\{\frac{rsin(t)-rf'(x_0)cos(t)}{\sqrt{1+[f'(x_0)]^2}}\right\}^2+\left\{\frac{rf'(x_0)sin(t)+rcos(t)}{\sqrt{1+[f'(x_0)]^2}}\right\}^2$$

$$\left\{x(t)-\left[x_0-\frac{rf'(x_0)}{\sqrt{1+[f'(x_0)]^2}}\right]\right\}^2+\left\{y(t)-\left[f(x_0)+\frac{r}{\sqrt{1+[f'(x_0)]^2}}\right]\right\}^2=$$

$$\frac{r^2sin^2(t)-2r^2f'(x_0)sin(t)cos(t)+r^2[f'(x_0)]^2cos^2(t)}{1+[f'(x_0)]^2}$$

$$+\frac{r^2[f'(x_0)]^2sin^2(t)+2r^2f'(x_0)sin(t)cos(t)+r^2cos^2(t)}{1+[f'(x_0)]^2}$$

$$\left\{x(t)-\left[x_0-\frac{rf'(x_0)}{\sqrt{1+[f'(x_0)]^2}}\right]\right\}^2+\left\{y(t)-\left[f(x_0)+\frac{r}{\sqrt{1+[f'(x_0)]^2}}\right]\right\}^2=$$

$$\frac{r^2sin^2(t)+r^2[f'(x_0)]^2cos^2(t)}{1+[f'(x_0)]^2}+\frac{r^2[f'(x_0)]^2sin^2(t)+r^2cos^2(t)}{1+[f'(x_0)]^2}$$

$$\left\{x(t)-\left[x_0-\frac{rf'(x_0)}{\sqrt{1+[f'(x_0)]^2}}\right]\right\}^2+\left\{y(t)-\left[f(x_0)+\frac{r}{\sqrt{1+[f'(x_0)]^2}}\right]\right\}^2=$$

$$\frac{r^2sin^2(t)+r^2cos^2(t)+r^2[f'(x_0)]^2sin^2(t)+r^2[f'(x_0)]^2cos^2(t)}{1+[f'(x_0)]^2}$$

$$\left\{x(t)-\left[x_0-\frac{rf'(x_0)}{\sqrt{1+[f'(x_0)]^2}}\right]\right\}^2+\left\{y(t)-\left[f(x_0)+\frac{r}{\sqrt{1+[f'(x_0)]^2}}\right]\right\}^2=$$

$$\frac{r^2+r^2[f'(x_0)]^2}{1+[f'(x_0)]^2}$$

Replacing $x(t)$ and $y(t)$ by x and y:

$$\left\{x-\left[x_0-\frac{rf'(x_0)}{\sqrt{1+[f'(x_0)]^2}}\right]\right\}^2+\left\{y-\left[f(x_0)+\frac{r}{\sqrt{1+[f'(x_0)]^2}}\right]\right\}^2=r^2$$

A2. Addition of the square of the expressions for the second tangent circle with radius r to the graph of a function f at the point $P(x_0, f(x_0))$ (Gómez-Villarraga, F. 2021)

$$\left\{x(t)-\left[x_0+\frac{rf'(x_0)}{\sqrt{1+[f'(x_0)]^2}}\right]\right\}^2+\left\{y(t)-\left[f(x_0)-\frac{r}{\sqrt{1+[f'(x_0)]^2}}\right]\right\}^2=$$

$$\left\{\frac{-rsin(t)+rf'(x_0)cos(t)}{\sqrt{1+[f'(x_0)]^2}}\right\}^2+\left\{\frac{-rf'(x_0)sin(t)-rcos(t)}{\sqrt{1+[f'(x_0)]^2}}\right\}^2$$

$$\left\{x(t)-\left[x_0+\frac{rf'(x_0)}{\sqrt{1+[f'(x_0)]^2}}\right]\right\}^2+\left\{y(t)-\left[f(x_0)-\frac{r}{\sqrt{1+[f'(x_0)]^2}}\right]\right\}^2=$$

$$\frac{r^2sin^2(t)-2r^2f'(x_0)sin(t)cos(t)+r^2[f'(x_0)]^2cos^2(t)}{1+[f'(x_0)]^2}$$

$$+\frac{r^2[f'(x_0)]^2sin^2(t)+2r^2f'(x_0)sin(t)cos(t)+r^2cos^2(t)}{1+[f'(x_0)]^2}$$

$$\left\{x(t)-\left[x_0+\frac{rf'(x_0)}{\sqrt{1+[f'(x_0)]^2}}\right]\right\}^2+\left\{y(t)-\left[f(x_0)-\frac{r}{\sqrt{1+[f'(x_0)]^2}}\right]\right\}^2=$$

$$\frac{r^2sin^2(t)+r^2[f'(x_0)]^2cos^2(t)}{1+[f'(x_0)]^2}+\frac{r^2[f'(x_0)]^2sin^2(t)+r^2cos^2(t)}{1+[f'(x_0)]^2}$$

$$\left\{x(t)-\left[x_0+\frac{rf'(x_0)}{\sqrt{1+[f'(x_0)]^2}}\right]\right\}^2+\left\{y(t)-\left[f(x_0)-\frac{r}{\sqrt{1+[f'(x_0)]^2}}\right]\right\}^2=$$

$$\frac{r^2sin^2(t)+r^2cos^2(t)+r^2[f'(x_0)]^2sin^2(t)+r^2[f'(x_0)]^2cos^2(t)}{1+[f'(x_0)]^2}$$

$$\left\{x(t)-\left[x_0+\frac{rf'(x_0)}{\sqrt{1+[f'(x_0)]^2}}\right]\right\}^2+\left\{y(t)-\left[f(x_0)-\frac{r}{\sqrt{1+[f'(x_0)]^2}}\right]\right\}^2=$$

$$\frac{r^2+r^2[f'(x_0)]^2}{1+[f'(x_0)]^2}$$

Replacing $x(t)$ and $y(t)$ by x and y:

$$\left\{x-\left[x_0+\frac{rf'(x_0)}{\sqrt{1+[f'(x_0)]^2}}\right]\right\}^2+\left\{y-\left[f(x_0)-\frac{r}{\sqrt{1+[f'(x_0)]^2}}\right]\right\}^2=r^2$$

Appendix B. Solution of the systems of equations for the case a) Three lines (LLL)

B.1. The tangent circle $\mathbb{C}_1$

$$x_{\mathbb{C}_1 l_1}+\frac{\mathbb{R}_1}{\sqrt{2}}=x_{\mathbb{C}_1 l_2}+\frac{\mathbb{R}_1}{\sqrt{2}} \quad \textbf{(1)}$$

$$x_{\mathbb{C}_1 l_2}+\frac{\mathbb{R}_1}{\sqrt{2}}=x_{\mathbb{C}_1 l_3}-5\frac{\mathbb{R}_1}{\sqrt{26}} \quad \textbf{(2)}$$

$$x_{\mathbb{C}_1 l_1} + 3 - \frac{\mathbb{R}_1}{\sqrt{2}} = -x_{\mathbb{C}_1 l_2} + 1 + \frac{\mathbb{R}_1}{\sqrt{2}} \quad \textbf{(3)}$$

$$-x_{\mathbb{C}_1 l_2} + 1 + \frac{\mathbb{R}_1}{\sqrt{2}} = -5x_{\mathbb{C}_1 l_3} + 18 - \frac{\mathbb{R}_1}{\sqrt{26}} \quad \textbf{(4)}$$

From the equation 1:

$$x_{\mathbb{C}_1 l_1} = x_{\mathbb{C}_1 l_2} \quad \textbf{(5)}$$

Replacing the equation 5 in the equation 3:

$$x_{\mathbb{C}_1 l_2} + 3 - \frac{\mathbb{R}_1}{\sqrt{2}} = -x_{\mathbb{C}_1 l_2} + 1 + \frac{\mathbb{R}_1}{\sqrt{2}}$$

$$2x_{\mathbb{C}_1 l_2} = -2 + 2\frac{\mathbb{R}_1}{\sqrt{2}}$$

$$x_{\mathbb{C}_1 l_2} = \frac{\mathbb{R}_1}{\sqrt{2}} - 1 \quad \textbf{(6)}$$

Replacing the equation 6 in the equation 2:

$$\frac{\mathbb{R}_1}{\sqrt{2}} - 1 + \frac{\mathbb{R}_1}{\sqrt{2}} = x_{\mathbb{C}_1 l_3} - 5\frac{\mathbb{R}_1}{\sqrt{26}}$$

$$x_{\mathbb{C}_1 l_3} = 2\frac{\mathbb{R}_1}{\sqrt{2}} + 5\frac{\mathbb{R}_1}{\sqrt{26}} - 1 \quad \textbf{(7)}$$

Replacing the equations 6 and 7 in the equation 4:

$$-\frac{\mathbb{R}_1}{\sqrt{2}} + 1 + 1 + \frac{\mathbb{R}_1}{\sqrt{2}} = -5\left(2\frac{\mathbb{R}_1}{\sqrt{2}} + 5\frac{\mathbb{R}_1}{\sqrt{26}} - 1\right) + 18 - \frac{\mathbb{R}_1}{\sqrt{26}}$$

$$2 = -10\frac{\mathbb{R}_1}{\sqrt{2}} - 25\frac{\mathbb{R}_1}{\sqrt{26}} + 5 + 18 - \frac{\mathbb{R}_1}{\sqrt{26}}$$

$$-21 = -10\frac{\mathbb{R}_1}{\sqrt{2}} - 26\frac{\mathbb{R}_1}{\sqrt{26}}$$

$$-21 = -5\mathbb{R}_1\sqrt{2} - \mathbb{R}_1\sqrt{26}$$

$$21 = \mathbb{R}_1\left(5\sqrt{2} + \sqrt{26}\right)$$

$$\mathbb{R}_1 = \frac{21}{5\sqrt{2}+\sqrt{26}} \quad \textbf{(8)}$$

Replacing the equation 8 in the equation 7:

$$x_{\mathbb{C}_1 l_3} = 2\frac{\frac{21}{5\sqrt{2}+\sqrt{26}}}{\sqrt{2}} + 5\frac{\frac{21}{5\sqrt{2}+\sqrt{26}}}{\sqrt{26}} - 1$$

$$x_{\mathbb{C}_1 l_3} = \frac{42}{10 + 2\sqrt{13}} + \frac{105}{10\sqrt{13} + 26} - 1$$

$$x_{\mathbb{C}_1 l_3} = \frac{21}{5+\sqrt{13}} + \frac{105}{10\sqrt{13}+26} - 1 \quad \textbf{(9)}$$

Replacing the equation 8 in the equation 6:

$$x_{\mathbb{C}_1 l_2} = \frac{\frac{21}{5\sqrt{2} + \sqrt{26}}}{\sqrt{2}} - 1$$

$$x_{\mathbb{C}_1 l_2} = \frac{21}{10+2\sqrt{13}} - 1 \quad \textbf{(10)}$$

Replacing the equation 10 in the equation 5:

$$x_{\mathbb{C}_1 l_1} = \frac{21}{10+2\sqrt{13}} - 1 \quad \textbf{(11)}$$

The equations 20, 24 and 28 (main text) have the format $\left(x - h_{\mathbb{C}_1}\right)^2 + \left(y - k_{\mathbb{C}_1}\right)^2 = \mathbb{R}_1^2$. $h_{\mathbb{C}_1}$ and $k_{\mathbb{C}_1}$ can be calculated from any of these equations. Taking the equation 20:

$$h_{\mathbb{C}_1} = x_{\mathbb{C}_1 l_1} + \frac{\mathbb{R}_1}{\sqrt{2}} \quad \textbf{(12)}$$

$$k_{\mathbb{C}_1} = x_{\mathbb{C}_1 l_1} + 3 - \frac{\mathbb{R}_1}{\sqrt{2}} \quad \textbf{(13)}$$

Replacing the equations 8 and 11 in the equation 12:

$$h_{\mathbb{C}_1} = \frac{21}{10 + 2\sqrt{13}} - 1 + \frac{\frac{21}{5\sqrt{2} + \sqrt{26}}}{\sqrt{2}}$$

$$h_{\mathbb{C}_1} = \frac{21}{10+2\sqrt{13}} - 1 + \frac{21}{10+2\sqrt{13}} = \frac{21}{5+\sqrt{13}} - 1 \quad \textbf{(14)}$$

Replacing the equations 8 and 11 in the equation 13:

$$k_{\mathbb{C}_1} = \frac{21}{10 + 2\sqrt{13}} - 1 + 3 - \frac{\frac{21}{5\sqrt{2} + \sqrt{26}}}{\sqrt{2}}$$

$$k_{\mathbb{C}_1} = \frac{21}{10+2\sqrt{13}} - 1 + 3 - \frac{21}{10+2\sqrt{13}} = 2 \quad \textbf{(15)}$$

B.2. The tangent circle $\mathbb{C}_2$

$$x_{\mathbb{C}_2 l_1} + \frac{\mathbb{R}_2}{\sqrt{2}} = x_{\mathbb{C}_2 l_2} + \frac{\mathbb{R}_2}{\sqrt{2}} \quad \textbf{(1)}$$

$$x_{\mathbb{C}_2 l_2} + \frac{\mathbb{R}_2}{\sqrt{2}} = x_{\mathbb{C}_2 l_3} + 5\frac{\mathbb{R}_2}{\sqrt{26}} \quad \textbf{(2)}$$

$$x_{\mathbb{C}_2 l_1} + 3 - \frac{\mathbb{R}_2}{\sqrt{2}} = -x_{\mathbb{C}_2 l_2} + 1 + \frac{\mathbb{R}_2}{\sqrt{2}} \quad \textbf{(3)}$$

$$-x_{\mathbb{C}_2 l_2} + 1 + \frac{\mathbb{R}_2}{\sqrt{2}} = -5x_{\mathbb{C}_2 l_3} + 18 + \frac{\mathbb{R}_2}{\sqrt{26}} \quad \textbf{(4)}$$

From the equation 1:

$$x_{\mathbb{C}_2 l_1} = x_{\mathbb{C}_2 l_2} \quad \textbf{(5)}$$

Replacing the equation 5 in the equation 3:

$$x_{\mathbb{C}_2 l_2} + 3 - \frac{\mathbb{R}_2}{\sqrt{2}} = -x_{\mathbb{C}_2 l_2} + 1 + \frac{\mathbb{R}_2}{\sqrt{2}}$$

$$2x_{\mathbb{C}_2 l_2} = 2\frac{\mathbb{R}_2}{\sqrt{2}} - 2$$

$$x_{\mathbb{C}_2 l_2} = \frac{\mathbb{R}_2}{\sqrt{2}} - 1 \quad \textbf{(6)}$$

Replacing the equation 6 in the equation 2:

$$\frac{\mathbb{R}_2}{\sqrt{2}} - 1 + \frac{\mathbb{R}_2}{\sqrt{2}} = x_{\mathbb{C}_2 l_3} + 5\frac{\mathbb{R}_2}{\sqrt{26}}$$

$$x_{\mathbb{C}_2 l_3} = 2\frac{\mathbb{R}_2}{\sqrt{2}} - 5\frac{\mathbb{R}_2}{\sqrt{26}} - 1 \quad \textbf{(7)}$$

Replacing the equations 6 and 7 in the equation 4:

$$1 - \frac{\mathbb{R}_2}{\sqrt{2}} + 1 + \frac{\mathbb{R}_2}{\sqrt{2}} = -5\left(2\frac{\mathbb{R}_2}{\sqrt{2}} - 5\frac{\mathbb{R}_2}{\sqrt{26}} - 1\right) + 18 + \frac{\mathbb{R}_2}{\sqrt{26}}$$

$$2 = -10\frac{\mathbb{R}_2}{\sqrt{2}} + 25\frac{\mathbb{R}_2}{\sqrt{26}} + 5 + 18 + \frac{\mathbb{R}_2}{\sqrt{26}}$$

$$-21 = -10\frac{\mathbb{R}_2}{\sqrt{2}} + 26\frac{\mathbb{R}_2}{\sqrt{26}}$$

$$-21 = -5\mathbb{R}_2\sqrt{2} + \mathbb{R}_2\sqrt{26}$$

$$21 = \mathbb{R}_2\left(5\sqrt{2} - \sqrt{26}\right)$$

$$\mathbb{R}_2 = \frac{21}{5\sqrt{2} - \sqrt{26}} \quad \textbf{(8)}$$

Replacing the equation 8 in the equation 7:

$$x_{\mathbb{C}_2 l_3} = 2\frac{\frac{21}{5\sqrt{2} - \sqrt{26}}}{\sqrt{2}} - 5\frac{\frac{21}{5\sqrt{2} - \sqrt{26}}}{\sqrt{26}} - 1$$

$$x_{\mathbb{C}_2 l_3} = \frac{42}{10 - 2\sqrt{13}} - \frac{105}{10\sqrt{13} - 26} - 1$$

$$x_{\mathbb{C}_2 l_3} = \frac{21}{5-\sqrt{13}} - \frac{105}{10\sqrt{13}-26} - 1 \quad \textbf{(9)}$$

Replacing the equation 8 in the equation 6:

$$x_{\mathbb{C}_2 l_2} = \frac{\frac{21}{5\sqrt{2}-\sqrt{26}}}{\sqrt{2}} - 1$$

$$x_{\mathbb{C}_2 l_2} = \frac{21}{10-2\sqrt{13}} - 1 \quad \textbf{(10)}$$

Replacing the equation 10 in the equation 5:

$$x_{\mathbb{C}_2 l_1} = \frac{21}{10-2\sqrt{13}} - 1 \quad \textbf{(11)}$$

The equations 42, 46 and 50 (main text) have the format $\left(x - h_{\mathbb{C}_2}\right)^2 + \left(y - k_{\mathbb{C}_2}\right)^2 = \mathbb{R}_2^2$. $h_{\mathbb{C}_2}$ and $k_{\mathbb{C}_2}$ can be calculated from any of these equations. Taking the equation 42:

$$h_{\mathbb{C}_2} = x_{\mathbb{C}_2 l_1} + \frac{\mathbb{R}_2}{\sqrt{2}} \quad \textbf{(12)}$$

$$k_{\mathbb{C}_2} = x_{\mathbb{C}_2 l_1} + 3 - \frac{\mathbb{R}_2}{\sqrt{2}} \quad \textbf{(13)}$$

Replacing the equations 8 and 11 in the equation 12:

$$h_{\mathbb{C}_2} = \frac{21}{10-2\sqrt{13}} - 1 + \frac{\frac{21}{5\sqrt{2}-\sqrt{26}}}{\sqrt{2}}$$

$$h_{\mathbb{C}_2} = \frac{21}{10-2\sqrt{13}} - 1 + \frac{21}{10-2\sqrt{13}} = \frac{21}{5-\sqrt{13}} - 1 \quad \textbf{(14)}$$

Replacing the equations 8 and 11 in the equation 13:

$$k_{\mathbb{C}_2} = \frac{21}{10-2\sqrt{13}} - 1 + 3 - \frac{\frac{21}{5\sqrt{2}-\sqrt{26}}}{\sqrt{2}}$$

$$k_{\mathbb{C}_2} = \frac{21}{10-2\sqrt{13}} - 1 + 3 - \frac{21}{10-2\sqrt{13}} = 2 \quad \textbf{(15)}$$

B.3. The tangent circle $\mathbb{C}_3$

$$x_{\mathbb{C}_3 l_1} - \frac{\mathbb{R}_3}{\sqrt{2}} = x_{\mathbb{C}_3 l_2} + \frac{\mathbb{R}_3}{\sqrt{2}} \quad \textbf{(1)}$$

$$x_{\mathbb{C}_3 l_2} + \frac{\mathbb{R}_3}{\sqrt{2}} = x_{\mathbb{C}_3 l_3} - 5\frac{\mathbb{R}_3}{\sqrt{26}} \quad \textbf{(2)}$$

$$x_{\mathbb{C}_3 l_1} + 3 + \frac{\mathbb{R}_3}{\sqrt{2}} = -x_{\mathbb{C}_3 l_2} + 1 + \frac{\mathbb{R}_3}{\sqrt{2}} \quad \textbf{(3)}$$

$$-x_{\mathbb{C}_3 l_2} + 1 + \frac{\mathbb{R}_3}{\sqrt{2}} = -5x_{\mathbb{C}_3 l_3} + 18 - \frac{\mathbb{R}_3}{\sqrt{26}} \quad \textbf{(4)}$$

From the equation 1:

$$x_{\mathbb{C}_3 l_1} = x_{\mathbb{C}_3 l_2} + 2\frac{\mathbb{R}_3}{\sqrt{2}} \quad \textbf{(5)}$$

Replacing the equation 5 in the equation 3:

$$x_{\mathbb{C}_3 l_2} + 2\frac{\mathbb{R}_3}{\sqrt{2}} + 3 + \frac{\mathbb{R}_3}{\sqrt{2}} = -x_{\mathbb{C}_3 l_2} + 1 + \frac{\mathbb{R}_3}{\sqrt{2}}$$

$$2x_{\mathbb{C}_3 l_2} = -2 - 2\frac{\mathbb{R}_3}{\sqrt{2}}$$

$$x_{\mathbb{C}_3 l_2} = -1 - \frac{\mathbb{R}_3}{\sqrt{2}} \quad \textbf{(6)}$$

Replacing the equation 6 in the equation 2:

$$-1 - \frac{\mathbb{R}_3}{\sqrt{2}} + \frac{\mathbb{R}_3}{\sqrt{2}} = x_{\mathbb{C}_3 l_3} - 5\frac{\mathbb{R}_3}{\sqrt{26}}$$

$$x_{\mathbb{C}_3 l_3} = -1 + 5\frac{\mathbb{R}_3}{\sqrt{26}} \quad \textbf{(7)}$$

Replacing the equations 6 and 7 in the equation 4:

$$1 + \frac{\mathbb{R}_3}{\sqrt{2}} + 1 + \frac{\mathbb{R}_3}{\sqrt{2}} = -5\left(-1 + 5\frac{\mathbb{R}_3}{\sqrt{26}}\right) + 18 - \frac{\mathbb{R}_3}{\sqrt{26}}$$

$$2 + 2\frac{\mathbb{R}_3}{\sqrt{2}} = 5 - 25\frac{\mathbb{R}_3}{\sqrt{26}} + 18 - \frac{\mathbb{R}_3}{\sqrt{26}}$$

$$-21 = -26\frac{\mathbb{R}_3}{\sqrt{26}} - 2\frac{\mathbb{R}_3}{\sqrt{2}}$$

$$-21 = -\mathbb{R}_3\sqrt{26} - \mathbb{R}_3\sqrt{2}$$

$$21 = \mathbb{R}_3\left(\sqrt{26} + \sqrt{2}\right)$$

$$\mathbb{R}_3 = \frac{21}{\sqrt{26}+\sqrt{2}} \quad \textbf{(8)}$$

Replacing the equation 8 in the equation 7:

$$x_{\mathbb{C}_3 l_3} = -1 + 5\frac{\frac{21}{\sqrt{26}+\sqrt{2}}}{\sqrt{26}}$$

$$x_{\mathbb{C}_3 l_3} = -1 + \frac{105}{26+2\sqrt{13}} \quad \textbf{(9)}$$

Replacing the equation 8 in the equation 6:

$$x_{\mathbb{C}_3 l_2} = -1 - \frac{\frac{21}{\sqrt{26}+\sqrt{2}}}{\sqrt{2}}$$

$$x_{\mathbb{C}_3 l_2} = -1 - \frac{21}{2\sqrt{13}+2} \quad \textbf{(10)}$$

Replacing the equations 8 and 10 in the equation 5:

$$x_{\mathbb{C}_3 l_1} = -1 - \frac{21}{2\sqrt{13}+2} + 2\frac{\frac{21}{\sqrt{26}+\sqrt{2}}}{\sqrt{2}}$$

$$x_{\mathbb{C}_3 l_1} = -1 - \frac{21}{2\sqrt{13}+2} + \frac{21}{\sqrt{13}+1} \quad \textbf{(11)}$$

The equations 64, 68 and 72 (main text) have the format $\left(x - h_{\mathbb{C}_3}\right)^2 + \left(y - k_{\mathbb{C}_3}\right)^2 = \mathbb{R}_3^2$. $h_{\mathbb{C}_3}$ and $k_{\mathbb{C}_3}$ can be calculated from any of these equations. Taking the equation 64:

$$h_{\mathbb{C}_3} = x_{\mathbb{C}_3 l_1} - \frac{\mathbb{R}_3}{\sqrt{2}} \quad \textbf{(12)}$$

$$k_{\mathbb{C}_3} = x_{\mathbb{C}_3 l_1} + 3 + \frac{\mathbb{R}_3}{\sqrt{2}} \quad \textbf{(13)}$$

Replacing the equations 8 and 11 in the equation 12:

$$h_{\mathbb{C}_3} = -1 - \frac{21}{2\sqrt{13}+2} + \frac{21}{\sqrt{13}+1} - \frac{\frac{21}{\sqrt{26}+\sqrt{2}}}{\sqrt{2}}$$

$$h_{\mathbb{C}_3} = -1 - \frac{21}{2\sqrt{13}+2} + \frac{21}{\sqrt{13}+1} - \frac{21}{2\sqrt{13}+2} = \frac{21}{\sqrt{13}+1} - 1 - \frac{21}{\sqrt{13}+1} = -1 \quad \textbf{(14)}$$

Replacing the equations 8 and 11 in the equation 13:

$$k_{\mathbb{C}_3} = -1 - \frac{21}{2\sqrt{13}+2} + \frac{21}{\sqrt{13}+1} + 3 + \frac{\frac{21}{\sqrt{26}+\sqrt{2}}}{\sqrt{2}}$$

$$k_{\mathbb{C}_3} = -1 - \frac{21}{2\sqrt{13}+2} + \frac{21}{\sqrt{13}+1} + 3 + \frac{21}{2\sqrt{13}+2} = 2 + \frac{21}{\sqrt{13}+1} \quad \textbf{(15)}$$

B.4. The tangent circle $\mathbb{C}_4$

$$x_{\mathbb{C}_4 l_1} + \frac{\mathbb{R}_4}{\sqrt{2}} = x_{\mathbb{C}_4 l_2} - \frac{\mathbb{R}_4}{\sqrt{2}} \quad \textbf{(1)}$$

$$x_{\mathbb{C}_4 l_2} - \frac{\mathbb{R}_4}{\sqrt{2}} = x_{\mathbb{C}_4 l_3} - 5\frac{\mathbb{R}_4}{\sqrt{26}} \quad \textbf{(2)}$$

$$x_{\mathbb{C}_4 l_1} + 3 - \frac{\mathbb{R}_4}{\sqrt{2}} = -x_{\mathbb{C}_4 l_2} + 1 - \frac{\mathbb{R}_4}{\sqrt{2}} \quad \textbf{(3)}$$

$$-x_{\mathbb{C}_4 l_2} + 1 - \frac{\mathbb{R}_4}{\sqrt{2}} = -5x_{\mathbb{C}_4 l_3} + 18 - \frac{\mathbb{R}_4}{\sqrt{26}} \quad \textbf{(4)}$$

From the equation 1:

$$x_{\mathbb{C}_4 l_1} = x_{\mathbb{C}_4 l_2} - 2\frac{\mathbb{R}_4}{\sqrt{2}} \quad \textbf{(5)}$$

Replacing the equation 5 in the equation 3:

$$x_{\mathbb{C}_4 l_2} - 2\frac{\mathbb{R}_4}{\sqrt{2}} + 3 - \frac{\mathbb{R}_4}{\sqrt{2}} = -x_{\mathbb{C}_4 l_2} + 1 - \frac{\mathbb{R}_4}{\sqrt{2}}$$

$$2x_{\mathbb{C}_4 l_2} = -2 + 2\frac{\mathbb{R}_4}{\sqrt{2}}$$

$$x_{\mathbb{C}_4 l_2} = -1 + \frac{\mathbb{R}_4}{\sqrt{2}} \quad \textbf{(6)}$$

Replacing the equation 6 in the equation 2:

$$-1 + \frac{\mathbb{R}_4}{\sqrt{2}} - \frac{\mathbb{R}_4}{\sqrt{2}} = x_{\mathbb{C}_4 l_3} - 5\frac{\mathbb{R}_4}{\sqrt{26}}$$

$$x_{\mathbb{C}_4 l_3} = -1 + 5\frac{\mathbb{R}_4}{\sqrt{26}} \quad \textbf{(7)}$$

Replacing the equations 6 and 7 in the equation 4:

$$1 - \frac{\mathbb{R}_4}{\sqrt{2}} + 1 - \frac{\mathbb{R}_4}{\sqrt{2}} = -5\left(-1 + 5\frac{\mathbb{R}_4}{\sqrt{26}}\right) + 18 - \frac{\mathbb{R}_4}{\sqrt{26}}$$

$$2 - 2\frac{\mathbb{R}_4}{\sqrt{2}} = 5 - 25\frac{\mathbb{R}_4}{\sqrt{26}} + 18 - \frac{\mathbb{R}_4}{\sqrt{26}}$$

$$-21 = -26\frac{\mathbb{R}_4}{\sqrt{26}} + 2\frac{\mathbb{R}_4}{\sqrt{2}}$$

$$-21 = -\mathbb{R}_4\sqrt{26} + \mathbb{R}_4\sqrt{2}$$

$$21 = \mathbb{R}_4\left(\sqrt{26} - \sqrt{2}\right)$$

$$\mathbb{R}_4 = \frac{21}{\sqrt{26} - \sqrt{2}} \quad \textbf{(8)}$$

Replacing the equation 8 in the equation 7:

$$x_{\mathbb{C}_4 l_3} = -1 + 5\frac{\frac{21}{\sqrt{26} - \sqrt{2}}}{\sqrt{26}}$$

$$x_{\mathbb{C}_4 l_3} = -1 + \frac{105}{26 - 2\sqrt{13}} \quad \textbf{(9)}$$

Replacing the equation 8 in the equation 6:

$$x_{\mathbb{C}_4 l_2} = -1 + \frac{\frac{21}{\sqrt{26} - \sqrt{2}}}{\sqrt{2}}$$

$$x_{\mathbb{C}_4 l_2} = -1 + \frac{21}{2\sqrt{13}-2} \quad \textbf{(10)}$$

Replacing the equations 8 and 10 in the equation 5:

$$x_{\mathbb{C}_4 l_1} = -1 + \frac{21}{2\sqrt{13}-2} - 2\frac{\frac{21}{\sqrt{26}-\sqrt{2}}}{\sqrt{2}}$$

$$x_{\mathbb{C}_4 l_1} = -1 + \frac{21}{2\sqrt{13}-2} - \frac{21}{\sqrt{13}-1} \quad \textbf{(11)}$$

The equations 86, 90 and 94 (main text) have the format $\left(x - h_{\mathbb{C}_4}\right)^2 + \left(y - k_{\mathbb{C}_4}\right)^2 = \mathbb{R}_4^2$. $h_{\mathbb{C}_4}$ and $k_{\mathbb{C}_4}$ can be calculated from any of these equations. Taking the equation 86:

$$h_{\mathbb{C}_4} = x_{\mathbb{C}_4 l_1} + \frac{\mathbb{R}_4}{\sqrt{2}} \quad \textbf{(12)}$$

$$k_{\mathbb{C}_4} = x_{\mathbb{C}_4 l_1} + 3 - \frac{\mathbb{R}_4}{\sqrt{2}} \quad \textbf{(13)}$$

Replacing the equations 8 and 11 in the equation 12:

$$h_{\mathbb{C}_4} = -1 + \frac{21}{2\sqrt{13}-2} - \frac{21}{\sqrt{13}-1} + \frac{\frac{21}{\sqrt{26}-\sqrt{2}}}{\sqrt{2}}$$

$$h_{\mathbb{C}_4} = -1 + \frac{21}{2\sqrt{13}-2} - \frac{21}{\sqrt{13}-1} + \frac{21}{2\sqrt{13}-2} = -1 + \frac{21}{\sqrt{13}-1} - \frac{21}{\sqrt{13}-1} = -1 \quad \textbf{(14)}$$

Replacing the equations 8 and 11 in the equation 13:

$$k_{\mathbb{C}_4} = -1 + \frac{21}{2\sqrt{13}-2} - \frac{21}{\sqrt{13}-1} + 3 - \frac{\frac{21}{\sqrt{26}-\sqrt{2}}}{\sqrt{2}}$$

$$k_{\mathbb{C}_4} = -1 + \frac{21}{2\sqrt{13}-2} - \frac{21}{\sqrt{13}-1} + 3 - \frac{21}{2\sqrt{13}-2} = 2 - \frac{21}{\sqrt{13}-1} \quad \textbf{(15)}$$

Appendix C. Derivatives at the tangency points for the tangent circle $\mathbb{C}_1$

$f'(x) = y' = 1$ (line l_1)

$f'(x) = y' = -1$ (line l_2)

$f'(x) = y' = -5$ (line l_3)

$$x_{\mathbb{C}_1 l_1} = \frac{21}{10 + 2\sqrt{13}} - 1$$

$$x_{\mathbb{C}_1 l_2} = \frac{21}{10 + 2\sqrt{13}} - 1$$

$$x_{\mathbb{C}_1 l_3} = \frac{21}{5 + \sqrt{13}} + \frac{105}{10\sqrt{13} + 26} - 1$$

$$\mathbb{R}_1 = \frac{21}{5\sqrt{2}+\sqrt{26}}$$

$$h_{\mathbb{C}_1} = \frac{21}{5+\sqrt{13}} - 1$$

$$k_{\mathbb{C}_1} = 2$$

The equation of a circle with center $\left(h_{\mathbb{C}_1}, k_{\mathbb{C}_1}\right)$ and radius $\mathbb{R}_1$ is given by:

$\left(x - h_{\mathbb{C}_1}\right)^2 + \left(y - k_{\mathbb{C}_1}\right)^2 = \mathbb{R}_1^2$ (Leithold, L. 1998)

Where two functions can be obtained:

$$f(x) = \sqrt{\mathbb{R}_1^2 - \left(x - h_{\mathbb{C}_1}\right)^2} + k_{\mathbb{C}_1}$$

$$f'(x) = -\frac{x - h_{\mathbb{C}_1}}{\sqrt{\mathbb{R}_1^2 - \left(x - h_{\mathbb{C}_1}\right)^2}}$$

$$f(x) = -\sqrt{\mathbb{R}_1^2 - \left(x - h_{\mathbb{C}_1}\right)^2} + k_{\mathbb{C}_1}$$

$$f'(x) = \frac{x - h_{\mathbb{C}_1}}{\sqrt{\mathbb{R}_1^2 - \left(x - h_{\mathbb{C}_1}\right)^2}}$$

$$f'\left(x_{\mathbb{C}_1 l_1}\right) = -\frac{\frac{21}{10+2\sqrt{13}} - 1 - \frac{21}{5+\sqrt{13}} + 1}{\sqrt{\left(\frac{21}{5\sqrt{2}+\sqrt{26}}\right)^2 - \left(\frac{21}{10+2\sqrt{13}} - 1 - \frac{21}{5+\sqrt{13}} + 1\right)^2}}$$

$$f'\left(x_{\mathbb{C}_1 l_1}\right) = -\frac{\frac{21}{10+2\sqrt{13}} - \frac{21}{5+\sqrt{13}}}{\sqrt{\left(\frac{21}{5\sqrt{2}+\sqrt{26}}\right)^2 - \left(\frac{21}{10+2\sqrt{13}} - \frac{21}{5+\sqrt{13}}\right)^2}}$$

$$f'\left(x_{\mathbb{C}_1 l_1}\right) = -\frac{\frac{21\left(5+\sqrt{13}\right) - 21\left(10+2\sqrt{13}\right)}{\left(10+2\sqrt{13}\right)\left(5+\sqrt{13}\right)}}{\sqrt{\left(\frac{21}{5\sqrt{2}+\sqrt{26}}\right)^2 - \left(\frac{21\left(5+\sqrt{13}\right) - 21\left(10+2\sqrt{13}\right)}{\left(10+2\sqrt{13}\right)\left(5+\sqrt{13}\right)}\right)^2}}$$

$$f'\left(x_{\mathbb{C}_1 l_1}\right) = -\frac{\frac{105+21\sqrt{13}-210-42\sqrt{13}}{50+10\sqrt{13}+10\sqrt{13}+26}}{\sqrt{\left(\frac{21}{5\sqrt{2}+\sqrt{26}}\right)^2-\left(\frac{105+21\sqrt{13}-210-42\sqrt{13}}{50+10\sqrt{13}+10\sqrt{13}+26}\right)^2}}$$

$$f'\left(x_{\mathbb{C}_1 l_1}\right) = -\frac{\frac{-105-21\sqrt{13}}{76+20\sqrt{13}}}{\sqrt{\left(\frac{21}{5\sqrt{2}+\sqrt{26}}\right)^2-\left(\frac{-105-21\sqrt{13}}{76+20\sqrt{13}}\right)^2}}$$

$$f'\left(x_{\mathbb{C}_1 l_1}\right) = \frac{\frac{105+21\sqrt{13}}{76+20\sqrt{13}}}{\sqrt{\left(\frac{21}{5\sqrt{2}+\sqrt{26}}\right)^2-\left(\frac{105+21\sqrt{13}}{76+20\sqrt{13}}\right)^2}}$$

$$f'\left(x_{\mathbb{C}_1 l_1}\right) = \frac{\frac{105+21\sqrt{13}}{76+20\sqrt{13}}}{\sqrt{\frac{441}{50+20\sqrt{13}+26}-\frac{11025+4410\sqrt{13}+5733}{\left(76+20\sqrt{13}\right)^2}}}$$

$$f'\left(x_{\mathbb{C}_1 l_1}\right) = \frac{\frac{105+21\sqrt{13}}{76+20\sqrt{13}}}{\sqrt{\frac{441}{76+20\sqrt{13}}-\frac{16758+4410\sqrt{13}}{\left(76+20\sqrt{13}\right)^2}}}$$

$$f'\left(x_{\mathbb{C}_1 l_1}\right) = \frac{\frac{105+21\sqrt{13}}{76+20\sqrt{13}}}{\sqrt{\frac{441\left(76+20\sqrt{13}\right)-16758-4410\sqrt{13}}{\left(76+20\sqrt{13}\right)^2}}}$$

$$f'\left(x_{\mathbb{C}_1 l_1}\right) = \frac{\frac{105+21\sqrt{13}}{76+20\sqrt{13}}}{\sqrt{\frac{33516+8820\sqrt{13}-16758-4410\sqrt{13}}{\left(76+20\sqrt{13}\right)^2}}}$$

$$f'\left(x_{\mathbb{C}_1 l_1}\right) = \frac{\frac{105 + 21\sqrt{13}}{76 + 20\sqrt{13}}}{\sqrt{\frac{16758 + 4410\sqrt{13}}{\left(76 + 20\sqrt{13}\right)^2}}}$$

$$f'\left(x_{\mathbb{C}_1 l_1}\right) = \frac{\frac{105+21\sqrt{13}}{76+20\sqrt{13}}}{\sqrt{\frac{\left(105+21\sqrt{13}\right)^2}{\left(76+20\sqrt{13}\right)^2}}} = 1$$ derivative of the tangent circle $\mathbb{C}_1$ at the tangency point $x_{\mathbb{C}_1 l_1}$

$$f'\left(x_{\mathbb{C}_1 l_2}\right) = \frac{\frac{21}{10 + 2\sqrt{13}} - 1 - \frac{21}{5 + \sqrt{13}} + 1}{\sqrt{\left(\frac{21}{5\sqrt{2} + \sqrt{26}}\right)^2 - \left(\frac{21}{10 + 2\sqrt{13}} - 1 - \frac{21}{5 + \sqrt{13}} + 1\right)^2}}$$

$$f'\left(x_{\mathbb{C}_1 l_2}\right) = \frac{\frac{21}{10 + 2\sqrt{13}} - \frac{21}{5 + \sqrt{13}}}{\sqrt{\left(\frac{21}{5\sqrt{2} + \sqrt{26}}\right)^2 - \left(\frac{21}{10 + 2\sqrt{13}} - \frac{21}{5 + \sqrt{13}}\right)^2}}$$

$$f'\left(x_{\mathbb{C}_1 l_2}\right) = \frac{\frac{21\left(5 + \sqrt{13}\right) - 21\left(10 + 2\sqrt{13}\right)}{\left(10 + 2\sqrt{13}\right)\left(5 + \sqrt{13}\right)}}{\sqrt{\left(\frac{21}{5\sqrt{2} + \sqrt{26}}\right)^2 - \left(\frac{21\left(5 + \sqrt{13}\right) - 21\left(10 + 2\sqrt{13}\right)}{\left(10 + 2\sqrt{13}\right)\left(5 + \sqrt{13}\right)}\right)^2}}$$

$$f'\left(x_{\mathbb{C}_1 l_2}\right) = \frac{\frac{105 + 21\sqrt{13} - 210 - 42\sqrt{13}}{50 + 10\sqrt{13} + 10\sqrt{13} + 26}}{\sqrt{\left(\frac{21}{5\sqrt{2} + \sqrt{26}}\right)^2 - \left(\frac{105 + 21\sqrt{13} - 210 - 42\sqrt{13}}{50 + 10\sqrt{13} + 10\sqrt{13} + 26}\right)^2}}$$

$$f'\left(x_{\mathbb{C}_1 l_2}\right) = \frac{\frac{-105 - 21\sqrt{13}}{76 + 20\sqrt{13}}}{\sqrt{\left(\frac{21}{5\sqrt{2} + \sqrt{26}}\right)^2 - \left(\frac{-105 - 21\sqrt{13}}{76 + 20\sqrt{13}}\right)^2}}$$

$$f'\left(x_{\mathbb{C}_1 l_2}\right) = -\frac{\frac{105 + 21\sqrt{13}}{76 + 20\sqrt{13}}}{\sqrt{\left(\frac{21}{5\sqrt{2} + \sqrt{26}}\right)^2 - \left(\frac{105 + 21\sqrt{13}}{76 + 20\sqrt{13}}\right)^2}}$$

$$f'\left(x_{\mathbb{C}_1 l_2}\right) = -\frac{\frac{105+21\sqrt{13}}{76+20\sqrt{13}}}{\sqrt{\frac{441}{50+20\sqrt{13}+26} - \frac{11025+4410\sqrt{13}+5733}{\left(76+20\sqrt{13}\right)^2}}}$$

$$f'\left(x_{\mathbb{C}_1 l_2}\right) = -\frac{\frac{105+21\sqrt{13}}{76+20\sqrt{13}}}{\sqrt{\frac{441}{76+20\sqrt{13}} - \frac{16758+4410\sqrt{13}}{\left(76+20\sqrt{13}\right)^2}}}$$

$$f'\left(x_{\mathbb{C}_1 l_2}\right) = -\frac{\frac{105+21\sqrt{13}}{76+20\sqrt{13}}}{\sqrt{\frac{441\left(76+20\sqrt{13}\right) - 16758 - 4410\sqrt{13}}{\left(76+20\sqrt{13}\right)^2}}}$$

$$f'\left(x_{\mathbb{C}_1 l_2}\right) = -\frac{\frac{105+21\sqrt{13}}{76+20\sqrt{13}}}{\sqrt{\frac{33516+8820\sqrt{13} - 16758 - 4410\sqrt{13}}{\left(76+20\sqrt{13}\right)^2}}}$$

$$f'\left(x_{\mathbb{C}_1 l_2}\right) = -\frac{\frac{105+21\sqrt{13}}{76+20\sqrt{13}}}{\sqrt{\frac{16758+4410\sqrt{13}}{\left(76+20\sqrt{13}\right)^2}}}$$

$f'\left(x_{\mathbb{C}_1 l_2}\right) = -\frac{\frac{105+21\sqrt{13}}{76+20\sqrt{13}}}{\sqrt{\frac{\left(105+21\sqrt{13}\right)^2}{\left(76+20\sqrt{13}\right)^2}}} = -1$ derivative of the tangent circle $\mathbb{C}_1$ at the tangency point $x_{\mathbb{C}_1 l_2}$

$$f'\left(x_{\mathbb{C}_1 l_3}\right) = -\frac{\frac{21}{5+\sqrt{13}} + \frac{105}{10\sqrt{13}+26} - 1 - \frac{21}{5+\sqrt{13}} + 1}{\sqrt{\left(\frac{21}{5\sqrt{2}+\sqrt{26}}\right)^2 - \left(\frac{21}{5+\sqrt{13}} + \frac{105}{10\sqrt{13}+26} - 1 - \frac{21}{5+\sqrt{13}} + 1\right)^2}}$$

$$f'\left(x_{\mathbb{C}_1 l_3}\right) = -\frac{\frac{105}{10\sqrt{13}+26}}{\sqrt{\left(\frac{21}{5\sqrt{2}+\sqrt{26}}\right)^2 - \left(\frac{105}{10\sqrt{13}+26}\right)^2}}$$

$$f'\left(x_{\mathbb{C}_1 l_3}\right) = -\frac{\frac{105}{10\sqrt{13}+26}}{\sqrt{\frac{441}{50+20\sqrt{13}+26} - \frac{11025}{1300+520\sqrt{13}+676}}}$$

$$f'\left(x_{\mathbb{C}_1 l_3}\right) = -\frac{\frac{105}{10\sqrt{13}+26}}{\sqrt{\frac{441}{76+20\sqrt{13}} - \frac{11025}{1976+520\sqrt{13}}}}$$

$$f'\left(x_{\mathbb{C}_1 l_3}\right) = -\frac{\frac{105}{10\sqrt{13}+26}}{\sqrt{\frac{441\left(1976+520\sqrt{13}\right) - 11025\left(76+20\sqrt{13}\right)}{\left(76+20\sqrt{13}\right)\left(1976+520\sqrt{13}\right)}}}$$

$$f'\left(x_{\mathbb{C}_1 l_3}\right) = -\frac{\frac{105}{10\sqrt{13}+26}}{\sqrt{\frac{871416+229320\sqrt{13}-837900-220500\sqrt{13}}{150176+39520\sqrt{13}+39520\sqrt{13}+135200}}}$$

$f'\left(x_{\mathbb{C}_1 l_3}\right) = -\frac{\frac{105}{10\sqrt{13}+26}}{\sqrt{\frac{33516+8820\sqrt{13}}{285376+79040\sqrt{13}}}} = -5$ derivative of the tangent circle $\mathbb{C}_1$ at the tangency point $x_{\mathbb{C}_1 l_3}$

Appendix D. Solution of the system of equations for the case b) One line and two points (LPP)

$$\left\{1 - \left[x_{\mathbb{C}_1 l} - \frac{\mathbb{R}_1}{\sqrt{2}}\right]\right\}^2 + \left\{7 - \left[\left(x_{\mathbb{C}_1 l} - 5\right) + \frac{\mathbb{R}_1}{\sqrt{2}}\right]\right\}^2 = \mathbb{R}_1^2 \qquad \textbf{(1)}$$

$$\left\{6 - \left[x_{\mathbb{C}_1 l} - \frac{\mathbb{R}_1}{\sqrt{2}}\right]\right\}^2 + \left\{8 - \left[\left(x_{\mathbb{C}_1 l} - 5\right) + \frac{\mathbb{R}_1}{\sqrt{2}}\right]\right\}^2 = \mathbb{R}_1^2 \qquad \textbf{(2)}$$

Subtracting the equation 2 from the equation 1:

$$1 - 2\left[x_{\mathbb{C}_1 l} - \frac{\mathbb{R}_1}{\sqrt{2}}\right] + \left[x_{\mathbb{C}_1 l} - \frac{\mathbb{R}_1}{\sqrt{2}}\right]^2 - 36 + 12\left[x_{\mathbb{C}_1 l} - \frac{\mathbb{R}_1}{\sqrt{2}}\right] - \left[x_{\mathbb{C}_1 l} - \frac{\mathbb{R}_1}{\sqrt{2}}\right]^2$$

$$49 - 14\left[\left(x_{\mathbb{C}_1 l} - 5\right) + \frac{\mathbb{R}_1}{\sqrt{2}}\right] + \left[\left(x_{\mathbb{C}_1 l} - 5\right) + \frac{\mathbb{R}_1}{\sqrt{2}}\right]^2 - 64 + 16\left[\left(x_{\mathbb{C}_1 l} - 5\right) + \frac{\mathbb{R}_1}{\sqrt{2}}\right]$$

$$-\left[\left(x_{\mathbb{C}_1 l} - 5\right) + \frac{\mathbb{R}_1}{\sqrt{2}}\right]^2 = 0$$

$$-35+10\left[x_{\mathbb{C}_1 l}-\frac{\mathbb{R}_1}{\sqrt{2}}\right]-15+2\left[\left(x_{\mathbb{C}_1 l}-5\right)+\frac{\mathbb{R}_1}{\sqrt{2}}\right]=0$$

$$10x_{\mathbb{C}_1 l}-10\frac{\mathbb{R}_1}{\sqrt{2}}+2x_{\mathbb{C}_1 l}-10+2\frac{\mathbb{R}_1}{\sqrt{2}}=50$$

$$-8\frac{\mathbb{R}_1}{\sqrt{2}}=60-12x_{\mathbb{C}_1 l}$$

$$\mathbb{R}_1=\frac{\sqrt{2}\left(60-12x_{\mathbb{C}_1 l}\right)}{-8}=\frac{\sqrt{2}\left(15-3x_{\mathbb{C}_1 l}\right)}{-2}=\frac{\sqrt{2}\left(3x_{\mathbb{C}_1 l}-15\right)}{2} \quad \textbf{(3)}$$

Operating with the equation 1:

$$1-2\left[x_{\mathbb{C}_1 l}-\frac{\mathbb{R}_1}{\sqrt{2}}\right]+\left[x_{\mathbb{C}_1 l}-\frac{\mathbb{R}_1}{\sqrt{2}}\right]^2+49-14\left[\left(x_{\mathbb{C}_1 l}-5\right)+\frac{\mathbb{R}_1}{\sqrt{2}}\right]+\left[\left(x_{\mathbb{C}_1 l}-5\right)+\frac{\mathbb{R}_1}{\sqrt{2}}\right]^2=\mathbb{R}_1^2$$

$$50-2x_{\mathbb{C}_1 l}+2\frac{\mathbb{R}_1}{\sqrt{2}}+x_{\mathbb{C}_1 l}^2-2x_{\mathbb{C}_1 l}\frac{\mathbb{R}_1}{\sqrt{2}}+\frac{\mathbb{R}_1^2}{2}-14x_{\mathbb{C}_1 l}+70-14\frac{\mathbb{R}_1}{\sqrt{2}}+\left(x_{\mathbb{C}_1 l}-5\right)^2$$

$$+2\left(x_{\mathbb{C}_1 l}-5\right)\frac{\mathbb{R}_1}{\sqrt{2}}+\frac{\mathbb{R}_1^2}{2}=\mathbb{R}_1^2$$

$$120-16x_{\mathbb{C}_1 l}-12\frac{\mathbb{R}_1}{\sqrt{2}}+x_{\mathbb{C}_1 l}^2-2x_{\mathbb{C}_1 l}\frac{\mathbb{R}_1}{\sqrt{2}}+x_{\mathbb{C}_1 l}^2-10x_{\mathbb{C}_1 l}+25+2x_{\mathbb{C}_1 l}\frac{\mathbb{R}_1}{\sqrt{2}}-10\frac{\mathbb{R}_1}{\sqrt{2}}=0$$

$$145-26x_{\mathbb{C}_1 l}-22\frac{\mathbb{R}_1}{\sqrt{2}}+2x_{\mathbb{C}_1 l}^2=0 \quad \textbf{(4)}$$

Replacing the equation 3 in the equation 4:

$$145-26x_{\mathbb{C}_1 l}-22\frac{\frac{\sqrt{2}\left(3x_{\mathbb{C}_1 l}-15\right)}{2}}{\sqrt{2}}+2x_{\mathbb{C}_1 l}^2=0$$

$$145-26x_{\mathbb{C}_1 l}-11\left(3x_{\mathbb{C}_1 l}-15\right)+2x_{\mathbb{C}_1 l}^2=0$$

$$145-26x_{\mathbb{C}_1 l}-33x_{\mathbb{C}_1 l}+165+2x_{\mathbb{C}_1 l}^2=0$$

$$2x_{\mathbb{C}_1 l}^2-59x_{\mathbb{C}_1 l}+310=0 \quad \textbf{(5)}$$

Solving the quadratic equation 5:

$$x_{\mathbb{C}_1 l}=\frac{59\pm\sqrt{59^2-4(2)(310)}}{4}$$

$$x_{\mathbb{C}_1 l}=\frac{59\pm\sqrt{3481-2480}}{4}$$

$$x_{\mathbb{C}_1 l}=\frac{59\pm\sqrt{1001}}{4}$$

There are two solutions for $x_{\mathbb{C}_1 l}$ ($x_{\mathbb{C}_1 l+}$ and $x_{\mathbb{C}_1 l-}$):

$$x_{\mathbb{C}_1 l+} = \frac{59+\sqrt{1001}}{4} \quad \textbf{(6)}$$

$$x_{\mathbb{C}_1 l-} = \frac{59-\sqrt{1001}}{4} \quad \textbf{(7)}$$

Replacing the equations 6 and 7 in the equation 3. There are two solutions for $\mathbb{R}_1$ ($\mathbb{R}_{1+}$ and $\mathbb{R}_{1-}$):

$$\mathbb{R}_{1+} = \frac{\sqrt{2}\left[3\left(\frac{59+\sqrt{1001}}{4}\right)-15\right]}{2} \quad \textbf{(8)}$$

$$\mathbb{R}_{1-} = \frac{\sqrt{2}\left[3\left(\frac{59-\sqrt{1001}}{4}\right)-15\right]}{2} \quad \textbf{(9)}$$

The equation 110 (main text) has the format $\left(x - h_{\mathbb{C}_1}\right)^2 + \left(y - k_{\mathbb{C}_1}\right)^2 = \mathbb{R}_1^2$. $h_{\mathbb{C}_1}$ and $k_{\mathbb{C}_1}$ can be calculated from this equation:

$$h_{\mathbb{C}_1} = x_{\mathbb{C}_1 l} - \frac{\mathbb{R}_1}{\sqrt{2}} \quad \textbf{(10)}$$

$$k_{\mathbb{C}_1} = \left(x_{\mathbb{C}_1 l} - 5\right) + \frac{\mathbb{R}_1}{\sqrt{2}} \quad \textbf{(11)}$$

There are two solutions for $h_{\mathbb{C}_1}$ ($h_{\mathbb{C}_1+}$ and $h_{\mathbb{C}_1-}$). Replacing the equations 6, 8, 7 and 9 in the equation 10:

$$h_{\mathbb{C}_1+} = \frac{59+\sqrt{1001}}{4} - \frac{\frac{\sqrt{2}\left[3\left(\frac{59+\sqrt{1001}}{4}\right)-15\right]}{2}}{\sqrt{2}} = \frac{59+\sqrt{1001}}{4} - \frac{3\left(\frac{59+\sqrt{1001}}{4}\right)-15}{2}$$

$$= \frac{-\left(\frac{59+\sqrt{1001}}{4}\right)+15}{2} = \frac{\frac{-59-\sqrt{1001}+60}{4}}{2} = \frac{1-\sqrt{1001}}{8}$$

$$h_{\mathbb{C}_1-} = \frac{59-\sqrt{1001}}{4} - \frac{\frac{\sqrt{2}\left[3\left(\frac{59-\sqrt{1001}}{4}\right)-15\right]}{2}}{\sqrt{2}} = \frac{59-\sqrt{1001}}{4} - \frac{3\left(\frac{59-\sqrt{1001}}{4}\right)-15}{2}$$

$$= \frac{-\left(\frac{59-\sqrt{1001}}{4}\right)+15}{2} = \frac{\frac{-59+\sqrt{1001}+60}{4}}{2} = \frac{1+\sqrt{1001}}{8}$$

There are two solutions for $k_{\mathbb{C}_1}$ ($k_{\mathbb{C}_1+}$ and $k_{\mathbb{C}_1-}$). Replacing the equations 6, 8, 7 and 9 in the equation 11:

$$k_{\mathbb{C}_1+} = \left(\frac{59+\sqrt{1001}}{4} - 5\right) + \frac{\frac{\sqrt{2}\left[3\left(\frac{59+\sqrt{1001}}{4}\right)-15\right]}{2}}{\sqrt{2}} = \frac{59+\sqrt{1001}}{4} - 5 + \frac{3\left(\frac{59+\sqrt{1001}}{4}\right)-15}{2}$$

$$= -5 + \frac{5\left(\frac{59+\sqrt{1001}}{4}\right) - 15}{2} = -5 + \frac{\frac{295+5\sqrt{1001}-60}{4}}{2} = -5 + \frac{235+5\sqrt{1001}}{8}$$

$$= \frac{235+5\sqrt{1001}-40}{8} = \frac{195+5\sqrt{1001}}{8}$$

$$k_{\mathbb{C}_{1-}} = \left(\frac{59-\sqrt{1001}}{4} - 5\right) + \frac{\frac{\sqrt{2}\left[3\left(\frac{59-\sqrt{1001}}{4}\right) - 15\right]}{2}}{\sqrt{2}} = \frac{59-\sqrt{1001}}{4} - 5 + \frac{3\left(\frac{59-\sqrt{1001}}{4}\right) - 15}{2}$$

$$= -5 + \frac{5\left(\frac{59-\sqrt{1001}}{4}\right) - 15}{2} = -5 + \frac{\frac{295-5\sqrt{1001}-60}{4}}{2} = -5 + \frac{235-5\sqrt{1001}}{8}$$

$$= \frac{235-5\sqrt{1001}-40}{8} = \frac{195-5\sqrt{1001}}{8}$$

Appendix E. Derivative at the tangency point for the tangent circle $\mathbb{C}_{1+}$

$$f'(x) = y' = 1 \qquad \text{(line } l\text{)}$$

$$x_{\mathbb{C}_1 l+} = \frac{59+\sqrt{1001}}{4}$$

$$\mathbb{R}_{1+} = \frac{\sqrt{2}\left[3\left(\frac{59+\sqrt{1001}}{4}\right) - 15\right]}{2}$$

$$h_{\mathbb{C}_{1+}} = \frac{1-\sqrt{1001}}{8}$$

$$k_{\mathbb{C}_{1+}} = \frac{195+5\sqrt{1001}}{8}$$

The equation of a circle with center $\left(h_{\mathbb{C}_{1+}}, k_{\mathbb{C}_{1+}}\right)$ and radius $\mathbb{R}_{1+}$ is given by:

$$\left(x - h_{\mathbb{C}_{1+}}\right)^2 + \left(y - k_{\mathbb{C}_{1+}}\right)^2 = \mathbb{R}_{1+}^2$$

Where two functions can be obtained:

$$f(x) = \sqrt{\mathbb{R}_{1+}^2 - \left(x - h_{\mathbb{C}_{1+}}\right)^2} + k_{\mathbb{C}_{1+}}$$

$$f'(x) = -\frac{x - h_{\mathbb{C}_{1+}}}{\sqrt{\mathbb{R}_{1+}^2 - \left(x - h_{\mathbb{C}_{1+}}\right)^2}}$$

$$f(x) = -\sqrt{\mathbb{R}_{1+}^2 - \left(x - h_{\mathbb{C}_{1+}}\right)^2} + k_{\mathbb{C}_{1+}}$$

$$f'(x)=\frac{x-h_{\mathbb{C}_{1+}}}{\sqrt{\mathbb{R}_{1+}^2-\left(x-h_{\mathbb{C}_{1+}}\right)^2}}$$

$$f'\left(x_{\mathbb{C}_1 l+}\right)=\frac{\frac{59+\sqrt{1001}}{4}-\frac{1-\sqrt{1001}}{8}}{\sqrt{\left\{\frac{\sqrt{2}\left[3\left(\frac{59+\sqrt{1001}}{4}\right)-15\right]}{2}\right\}^2-\left(\frac{59+\sqrt{1001}}{4}-\frac{1-\sqrt{1001}}{8}\right)^2}}$$

$$f'\left(x_{\mathbb{C}_1 l+}\right)=\frac{\frac{118+2\sqrt{1001}-1+\sqrt{1001}}{8}}{\sqrt{\frac{\left[3\left(\frac{59+\sqrt{1001}}{4}\right)-15\right]^2}{2}-\left(\frac{118+2\sqrt{1001}-1+\sqrt{1001}}{8}\right)^2}}$$

$$f'\left(x_{\mathbb{C}_1 l+}\right)=\frac{\frac{117+3\sqrt{1001}}{8}}{\sqrt{\frac{9\left(\frac{59+\sqrt{1001}}{4}\right)^2-90\left(\frac{59+\sqrt{1001}}{4}\right)+225}{2}-\left(\frac{117+3\sqrt{1001}}{8}\right)^2}}$$

$$f'\left(x_{\mathbb{C}_1 l+}\right)=\frac{\frac{117+3\sqrt{1001}}{8}}{\sqrt{\frac{9\left(\frac{3481+118\sqrt{1001}+1001}{16}\right)-\frac{5310+90\sqrt{1001}}{4}+225}{2}-\frac{13689+702\sqrt{1001}+9009}{64}}}$$

$$f'\left(x_{\mathbb{C}_1 l+}\right)=\frac{\frac{117+3\sqrt{1001}}{8}}{\sqrt{\frac{9\left(\frac{4482+118\sqrt{1001}}{16}\right)-\frac{5310+90\sqrt{1001}}{4}+225}{2}-\frac{22698+702\sqrt{1001}}{64}}}$$

$$f'\left(x_{\mathbb{C}_1 l+}\right)=\frac{\frac{117+3\sqrt{1001}}{8}}{\sqrt{\frac{\frac{40338+1062\sqrt{1001}}{16}-\frac{5310+90\sqrt{1001}}{4}+225}{2}-\frac{22698+702\sqrt{1001}}{64}}}$$

$$f'\left(x_{\mathbb{C}_1 l+}\right)=\frac{\frac{117+3\sqrt{1001}}{8}}{\sqrt{\frac{\frac{40338+1062\sqrt{1001}-21240-360\sqrt{1001}+3600}{16}}{2}-\frac{22698+702\sqrt{1001}}{64}}}$$

$$f'(x_{\mathbb{C}_1 l+}) = \frac{\frac{117+3\sqrt{1001}}{8}}{\sqrt{\frac{\frac{22698+702\sqrt{1001}}{16}}{2} - \frac{22698+702\sqrt{1001}}{64}}}$$

$$f'(x_{\mathbb{C}_1 l+}) = \frac{\frac{117+3\sqrt{1001}}{8}}{\sqrt{\frac{22698+702\sqrt{1001}}{32} - \frac{22698+702\sqrt{1001}}{64}}}$$

$$f'(x_{\mathbb{C}_1 l+}) = \frac{\frac{117+3\sqrt{1001}}{8}}{\sqrt{\frac{45396+1404\sqrt{1001}-22698-702\sqrt{1001}}{64}}}$$

$f'(x_{\mathbb{C}_1 l+}) = \frac{\frac{117+3\sqrt{1001}}{8}}{\sqrt{\frac{22698+702\sqrt{1001}}{64}}} = 1$ derivative of the tangent circle $\mathbb{C}_{1+}$at the tangency point $x_{\mathbb{C}_1 l+}$

Appendix F. Circle equations calculations for the case c) Three circles (CCC)

F.1. Tangent circle $\mathbb{C}_1$ calculated in relation to the given circle C_1

$$\left\{x - \left[x_{\mathbb{C}_1 C_1} + \frac{\frac{\mathbb{R}_1 x_{\mathbb{C}_1 C_1}}{\sqrt{4 - x^2_{\mathbb{C}_1 C_1}}}}{\sqrt{1 + \frac{x^2_{\mathbb{C}_1 C_1}}{4 - x^2_{\mathbb{C}_1 C_1}}}}\right]\right\}^2 + \left\{y - \left[\sqrt{4 - x^2_{\mathbb{C}_1 C_1}} + \frac{\mathbb{R}_1}{\sqrt{1 + \frac{x^2_{\mathbb{C}_1 C_1}}{4 - x^2_{\mathbb{C}_1 C_1}}}}\right]\right\}^2 = \mathbb{R}_1^2$$

$$\left\{x - \left[x_{\mathbb{C}_1 C_1} + \frac{\frac{\mathbb{R}_1 x_{\mathbb{C}_1 C_1}}{\sqrt{4 - x^2_{\mathbb{C}_1 C_1}}}}{\sqrt{\frac{4 - x^2_{\mathbb{C}_1 C_1} + x^2_{\mathbb{C}_1 C_1}}{4 - x^2_{\mathbb{C}_1 C_1}}}}\right]\right\}^2 + \left\{y - \left[\sqrt{4 - x^2_{\mathbb{C}_1 C_1}} + \frac{\mathbb{R}_1}{\sqrt{\frac{4 - x^2_{\mathbb{C}_1 C_1} + x^2_{\mathbb{C}_1 C_1}}{4 - x^2_{\mathbb{C}_1 C_1}}}}\right]\right\}^2 = \mathbb{R}_1^2$$

$$\left\{x - \left[x_{\mathbb{C}_1 C_1} + \frac{\frac{\mathbb{R}_1 x_{\mathbb{C}_1 C_1}}{\sqrt{4 - x^2_{\mathbb{C}_1 C_1}}}}{\sqrt{\frac{4}{4 - x^2_{\mathbb{C}_1 C_1}}}}\right]\right\}^2 + \left\{y - \left[\sqrt{4 - x^2_{\mathbb{C}_1 C_1}} + \frac{\mathbb{R}_1}{\sqrt{\frac{4}{4 - x^2_{\mathbb{C}_1 C_1}}}}\right]\right\}^2 = \mathbb{R}_1^2$$

$$\left\{x - \left[x_{\mathbb{C}_1 C_1} + \frac{\mathbb{R}_1 x_{\mathbb{C}_1 C_1}}{2}\right]\right\}^2 + \left\{y - \left[\sqrt{4 - x^2_{\mathbb{C}_1 C_1}} + \frac{\mathbb{R}_1\sqrt{4 - x^2_{\mathbb{C}_1 C_1}}}{2}\right]\right\}^2 = \mathbb{R}_1^2$$

$$\left\{x-\left[x_{\mathbb{C}_1C_1}+\frac{\mathbb{R}_1 x_{\mathbb{C}_1C_1}}{2}\right]\right\}^2+\left\{y-\left[\sqrt{4-x_{\mathbb{C}_1C_1}^2}\left(1+\frac{\mathbb{R}_1}{2}\right)\right]\right\}^2=\mathbb{R}_1^2$$

F.2. Tangent circle $\mathbb{C}_1$ calculated in relation to the given circle C_2

$$\left\{x-\left[x_{\mathbb{C}_1C_2}+\frac{\frac{\mathbb{R}_1\left(x_{\mathbb{C}_1C_2}-7\right)}{\sqrt{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2}}}{\sqrt{1+\frac{\left(x_{\mathbb{C}_1C_2}-7\right)^2}{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2}}}\right]\right\}^2+\left\{y-\left[\sqrt{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2}-1+\frac{\mathbb{R}_1}{\sqrt{1+\frac{\left(x_{\mathbb{C}_1C_2}-7\right)^2}{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2}}}\right]\right\}^2=\mathbb{R}_1^2$$

$$\left\{x-\left[x_{\mathbb{C}_1C_2}+\frac{\frac{\mathbb{R}_1\left(x_{\mathbb{C}_1C_2}-7\right)}{\sqrt{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2}}}{\sqrt{\frac{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2+\left(x_{\mathbb{C}_1C_2}-7\right)^2}{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2}}}\right]\right\}^2+\left\{y-\left[\sqrt{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2}-1+\frac{\mathbb{R}_1}{\sqrt{\frac{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2+\left(x_{\mathbb{C}_1C_2}-7\right)^2}{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2}}}\right]\right\}^2=\mathbb{R}_1^2$$

$$\left\{x-\left[x_{\mathbb{C}_1C_2}+\frac{\frac{\mathbb{R}_1\left(x_{\mathbb{C}_1C_2}-7\right)}{\sqrt{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2}}}{\sqrt{\frac{9}{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2}}}\right]\right\}^2+\left\{y-\left[\sqrt{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2}-1+\frac{\mathbb{R}_1}{\sqrt{\frac{9}{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2}}}\right]\right\}^2=\mathbb{R}_1^2$$

$$\left\{x-\left[x_{\mathbb{C}_1C_2}+\frac{\mathbb{R}_1\left(x_{\mathbb{C}_1C_2}-7\right)}{3}\right]\right\}^2+\left\{y-\left[\sqrt{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2}-1+\frac{\mathbb{R}_1\sqrt{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2}}{3}\right]\right\}^2=\mathbb{R}_1^2$$

$$\left\{x-\left[x_{\mathbb{C}_1C_2}+\frac{\mathbb{R}_1\left(x_{\mathbb{C}_1C_2}-7\right)}{3}\right]\right\}^2+\left\{y-\left[\sqrt{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2}\left(1+\frac{\mathbb{R}_1}{3}\right)-1\right]\right\}^2=\mathbb{R}_1^2$$

F.3. Tangent circle $\mathbb{C}_1$ calculated in relation to the given circle C_3

$$\left\{x-\left[x_{\mathbb{C}_1C_3}+\frac{\frac{\mathbb{R}_1\left(x_{\mathbb{C}_1C_3}-3\right)}{\sqrt{16-\left(x_{\mathbb{C}_1C_3}-3\right)^2}}}{\sqrt{1+\frac{\left(x_{\mathbb{C}_1C_3}-3\right)^2}{16-\left(x_{\mathbb{C}_1C_3}-3\right)^2}}}\right]\right\}^2+\left\{y-\left[-\sqrt{16-\left(x_{\mathbb{C}_1C_3}-3\right)^2}+10-\frac{\mathbb{R}_1}{\sqrt{1+\frac{\left(x_{\mathbb{C}_1C_3}-3\right)^2}{16-\left(x_{\mathbb{C}_1C_3}-3\right)^2}}}\right]\right\}^2=\mathbb{R}_1^2$$

$$\left\{x-\left[x_{\mathbb{C}_1C_3}+\frac{\frac{\mathbb{R}_1\left(x_{\mathbb{C}_1C_3}-3\right)}{\sqrt{16-\left(x_{\mathbb{C}_1C_3}-3\right)^2}}}{\sqrt{\frac{16-\left(x_{\mathbb{C}_1C_3}-3\right)^2+\left(x_{\mathbb{C}_1C_3}-3\right)^2}{16-\left(x_{\mathbb{C}_1C_3}-3\right)^2}}}\right]\right\}^2+\left\{y-\left[-\sqrt{16-\left(x_{\mathbb{C}_1C_3}-3\right)^2}+10-\frac{\mathbb{R}_1}{\sqrt{\frac{16-\left(x_{\mathbb{C}_1C_3}-3\right)^2+\left(x_{\mathbb{C}_1C_3}-3\right)^2}{16-\left(x_{\mathbb{C}_1C_3}-3\right)^2}}}\right]\right\}^2=\mathbb{R}_1^2$$

$$\left\{x-\left[x_{\mathbb{C}_1C_3}+\frac{\frac{\mathbb{R}_1(x_{\mathbb{C}_1C_3}-3)}{\sqrt{16-(x_{\mathbb{C}_1C_3}-3)^2}}}{\sqrt{\frac{16}{16-(x_{\mathbb{C}_1C_3}-3)^2}}}\right]\right\}^2+\left\{y-\left[-\sqrt{16-(x_{\mathbb{C}_1C_3}-3)^2}+10-\frac{\mathbb{R}_1}{\sqrt{\frac{16}{16-(x_{\mathbb{C}_1C_3}-3)^2}}}\right]\right\}^2=\mathbb{R}_1^2$$

$$\left\{x-\left[x_{\mathbb{C}_1C_3}+\frac{\mathbb{R}_1(x_{\mathbb{C}_1C_3}-3)}{4}\right]\right\}^2+\left\{y-\left[-\sqrt{16-(x_{\mathbb{C}_1C_3}-3)^2}+10-\frac{\mathbb{R}_1\sqrt{16-(x_{\mathbb{C}_1C_3}-3)^2}}{4}\right]\right\}^2=\mathbb{R}_1^2$$

$$\left\{x-\left[x_{\mathbb{C}_1C_3}+\frac{\mathbb{R}_1(x_{\mathbb{C}_1C_3}-3)}{4}\right]\right\}^2+\left\{y-\left[-\sqrt{16-(x_{\mathbb{C}_1C_3}-3)^2}\left(1+\frac{\mathbb{R}_1}{4}\right)+10\right]\right\}^2=\mathbb{R}_1^2$$

Appendix G. Solution of the system of equations for the case c) Three circles (CCC) (the tangent circle $\mathbb{C}_1$)

$$x_{\mathbb{C}_1C_1}+\frac{\mathbb{R}_1x_{\mathbb{C}_1C_1}}{2}=x_{\mathbb{C}_1C_2}+\frac{\mathbb{R}_1(x_{\mathbb{C}_1C_2}-7)}{3} \quad \textbf{(1)}$$

$$x_{\mathbb{C}_1C_2}+\frac{\mathbb{R}_1(x_{\mathbb{C}_1C_2}-7)}{3}=x_{\mathbb{C}_1C_3}+\frac{\mathbb{R}_1(x_{\mathbb{C}_1C_3}-3)}{4} \quad \textbf{(2)}$$

$$\sqrt{4-x_{\mathbb{C}_1C_1}^2}\left(1+\frac{\mathbb{R}_1}{2}\right)=\sqrt{9-(x_{\mathbb{C}_1C_2}-7)^2}\left(1+\frac{\mathbb{R}_1}{3}\right)-1 \quad \textbf{(3)}$$

$$\sqrt{9-(x_{\mathbb{C}_1C_2}-7)^2}\left(1+\frac{\mathbb{R}_1}{3}\right)-1=-\sqrt{16-(x_{\mathbb{C}_1C_3}-3)^2}\left(1+\frac{\mathbb{R}_1}{4}\right)+10 \quad \textbf{(4)}$$

From the equation 1:

$$x_{\mathbb{C}_1C_1}\left(1+\frac{\mathbb{R}_1}{2}\right)=x_{\mathbb{C}_1C_2}+\frac{\mathbb{R}_1(x_{\mathbb{C}_1C_2}-7)}{3}$$

$$x_{\mathbb{C}_1C_1}=\frac{x_{\mathbb{C}_1C_2}+\frac{\mathbb{R}_1(x_{\mathbb{C}_1C_2}-7)}{3}}{1+\frac{\mathbb{R}_1}{2}} \quad \textbf{(5)}$$

Squaring the equation 3:

$$\left[\sqrt{4-x_{\mathbb{C}_1C_1}^2}\left(1+\frac{\mathbb{R}_1}{2}\right)\right]^2=\left[\sqrt{9-(x_{\mathbb{C}_1C_2}-7)^2}\left(1+\frac{\mathbb{R}_1}{3}\right)-1\right]^2 \quad \textbf{(6)}$$

$$(4-x_{\mathbb{C}_1C_1}^2)\left(1+\frac{\mathbb{R}_1}{2}\right)^2=\left[9-(x_{\mathbb{C}_1C_2}-7)^2\right]\left(1+\frac{\mathbb{R}_1}{3}\right)^2-2\sqrt{9-(x_{\mathbb{C}_1C_2}-7)^2}\left(1+\frac{\mathbb{R}_1}{3}\right)+1$$

$$4\left(1+\frac{\mathbb{R}_1}{2}\right)^2-x_{\mathbb{C}_1C_1}^2\left(1+\frac{\mathbb{R}_1}{2}\right)^2=9\left(1+\frac{\mathbb{R}_1}{3}\right)^2-(x_{\mathbb{C}_1C_2}-7)^2\left(1+\frac{\mathbb{R}_1}{3}\right)^2-2\sqrt{9-(x_{\mathbb{C}_1C_2}-7)^2}\left(1+\frac{\mathbb{R}_1}{3}\right)+1$$

$$4\left(1+\frac{\mathbb{R}_1}{2}\right)^2-x_{\mathbb{C}_1C_1}^2\left(1+\frac{\mathbb{R}_1}{2}\right)^2=9\left(1+\frac{\mathbb{R}_1}{3}\right)^2-(x_{\mathbb{C}_1C_2}^2-14x_{\mathbb{C}_1C_2}+49)\left(1+\frac{\mathbb{R}_1}{3}\right)^2$$

$$-2\sqrt{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2}\left(1+\frac{\mathbb{R}_1}{3}\right)+1$$

$$4\left(1+\frac{\mathbb{R}_1}{2}\right)^2-x_{\mathbb{C}_1C_1}^2\left(1+\frac{\mathbb{R}_1}{2}\right)^2=9\left(1+\frac{\mathbb{R}_1}{3}\right)^2-x_{\mathbb{C}_1C_2}^2\left(1+\frac{\mathbb{R}_1}{3}\right)^2+14x_{\mathbb{C}_1C_2}\left(1+\frac{\mathbb{R}_1}{3}\right)^2-49\left(1+\frac{\mathbb{R}_1}{3}\right)^2$$

$$-2\sqrt{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2}\left(1+\frac{\mathbb{R}_1}{3}\right)+1$$

$$4\left(1+\frac{\mathbb{R}_1}{2}\right)^2-x_{\mathbb{C}_1C_1}^2\left(1+\frac{\mathbb{R}_1}{2}\right)^2=-x_{\mathbb{C}_1C_2}^2\left(1+\frac{\mathbb{R}_1}{3}\right)^2+14x_{\mathbb{C}_1C_2}\left(1+\frac{\mathbb{R}_1}{3}\right)^2-40\left(1+\frac{\mathbb{R}_1}{3}\right)^2$$

$$-2\sqrt{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2}\left(1+\frac{\mathbb{R}_1}{3}\right)+1 \qquad \textbf{(7)}$$

Replacing the equation 5 in the equation 7:

$$4\left(1+\frac{\mathbb{R}_1}{2}\right)^2-\left[\frac{x_{\mathbb{C}_1C_2}+\frac{\mathbb{R}_1\left(x_{\mathbb{C}_1C_2}-7\right)}{3}}{1+\frac{\mathbb{R}_1}{2}}\right]^2\left(1+\frac{\mathbb{R}_1}{2}\right)^2=-x_{\mathbb{C}_1C_2}^2\left(1+\frac{\mathbb{R}_1}{3}\right)^2+14x_{\mathbb{C}_1C_2}\left(1+\frac{\mathbb{R}_1}{3}\right)^2-40\left(1+\frac{\mathbb{R}_1}{3}\right)^2$$

$$-2\sqrt{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2}\left(1+\frac{\mathbb{R}_1}{3}\right)+1$$

$$4\left(1+\frac{\mathbb{R}_1}{2}\right)^2-\left[\frac{x_{\mathbb{C}_1C_2}^2}{\left(1+\frac{\mathbb{R}_1}{2}\right)^2}+\frac{\frac{2x_{\mathbb{C}_1C_2}\mathbb{R}_1\left(x_{\mathbb{C}_1C_2}-7\right)}{3}}{\left(1+\frac{\mathbb{R}_1}{2}\right)^2}+\frac{\frac{\mathbb{R}_1^2\left(x_{\mathbb{C}_1C_2}-7\right)^2}{9}}{\left(1+\frac{\mathbb{R}_1}{2}\right)^2}\right]\left(1+\frac{\mathbb{R}_1}{2}\right)^2=-x_{\mathbb{C}_1C_2}^2\left(1+\frac{\mathbb{R}_1}{3}\right)^2$$

$$+14x_{\mathbb{C}_1C_2}\left(1+\frac{\mathbb{R}_1}{3}\right)^2-40\left(1+\frac{\mathbb{R}_1}{3}\right)^2-2\sqrt{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2}\left(1+\frac{\mathbb{R}_1}{3}\right)+1$$

$$4\left(1+\frac{\mathbb{R}_1}{2}\right)^2-x_{\mathbb{C}_1C_2}^2-\frac{2x_{\mathbb{C}_1C_2}\mathbb{R}_1\left(x_{\mathbb{C}_1C_2}-7\right)}{3}-\frac{\mathbb{R}_1^2\left(x_{\mathbb{C}_1C_2}-7\right)^2}{9}=-x_{\mathbb{C}_1C_2}^2\left(1+\frac{\mathbb{R}_1}{3}\right)^2$$

$$+14x_{\mathbb{C}_1C_2}\left(1+\frac{\mathbb{R}_1}{3}\right)^2-40\left(1+\frac{\mathbb{R}_1}{3}\right)^2-2\sqrt{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2}\left(1+\frac{\mathbb{R}_1}{3}\right)+1$$

$$4\left(1+\frac{\mathbb{R}_1}{2}\right)^2-x_{\mathbb{C}_1C_2}^2-\frac{2x_{\mathbb{C}_1C_2}^2\mathbb{R}_1}{3}+\frac{14x_{\mathbb{C}_1C_2}\mathbb{R}_1}{3}-\frac{\mathbb{R}_1^2}{9}\left(x_{\mathbb{C}_1C_2}^2-14x_{\mathbb{C}_1C_2}+49\right)=-x_{\mathbb{C}_1C_2}^2\left(1+\frac{\mathbb{R}_1}{3}\right)^2$$

$$+14x_{\mathbb{C}_1C_2}\left(1+\frac{\mathbb{R}_1}{3}\right)^2-40\left(1+\frac{\mathbb{R}_1}{3}\right)^2-2\sqrt{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2}\left(1+\frac{\mathbb{R}_1}{3}\right)+1$$

$$4\left(1+\frac{\mathbb{R}_1}{2}\right)^2-x_{\mathbb{C}_1C_2}^2-\frac{2x_{\mathbb{C}_1C_2}^2\mathbb{R}_1}{3}+\frac{14x_{\mathbb{C}_1C_2}\mathbb{R}_1}{3}-\frac{\mathbb{R}_1^2}{9}x_{\mathbb{C}_1C_2}^2+\frac{14\mathbb{R}_1^2}{9}x_{\mathbb{C}_1C_2}-\frac{49\mathbb{R}_1^2}{9}=-x_{\mathbb{C}_1C_2}^2\left(1+\frac{\mathbb{R}_1}{3}\right)^2$$

$$+14x_{\mathbb{C}_1C_2}\left(1+\frac{\mathbb{R}_1}{3}\right)^2-40\left(1+\frac{\mathbb{R}_1}{3}\right)^2-2\sqrt{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2}\left(1+\frac{\mathbb{R}_1}{3}\right)+1$$

$$4\left(1+\frac{\mathbb{R}_1}{2}\right)^2+40\left(1+\frac{\mathbb{R}_1}{3}\right)^2-1-\frac{49\mathbb{R}_1^2}{9}-x_{\mathbb{C}_1C_2}^2\left(1+\frac{\mathbb{R}_1}{3}\right)^2+\frac{14x_{\mathbb{C}_1C_2}\mathbb{R}_1}{3}+\frac{14\mathbb{R}_1^2}{9}x_{\mathbb{C}_1C_2}-14x_{\mathbb{C}_1C_2}\left(1+\frac{\mathbb{R}_1}{3}\right)^2$$

$$=-x_{\mathbb{C}_1C_2}^2\left(1+\frac{\mathbb{R}_1}{3}\right)^2-2\sqrt{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2}\left(1+\frac{\mathbb{R}_1}{3}\right)$$

$$4\left(1+\frac{\mathbb{R}_1}{2}\right)^2+40\left(1+\frac{\mathbb{R}_1}{3}\right)^2-1-\frac{49\mathbb{R}_1^2}{9}+\frac{14x_{\mathbb{C}_1C_2}\mathbb{R}_1}{3}+\frac{14\mathbb{R}_1^2}{9}x_{\mathbb{C}_1C_2}-14x_{\mathbb{C}_1C_2}\left(1+\frac{2\mathbb{R}_1}{3}+\frac{\mathbb{R}_1^2}{9}\right)$$

$$= -2\sqrt{9 - \left(x_{\mathbb{C}_1 C_2} - 7\right)^2}\left(1 + \frac{\mathbb{R}_1}{3}\right)$$

$$4\left(1 + \frac{\mathbb{R}_1}{2}\right)^2 + 40\left(1 + \frac{\mathbb{R}_1}{3}\right)^2 - 1 - \frac{49\mathbb{R}_1^2}{9} + \frac{14x_{\mathbb{C}_1 C_2}\mathbb{R}_1}{3} + \frac{14\mathbb{R}_1^2}{9}x_{\mathbb{C}_1 C_2} - 14x_{\mathbb{C}_1 C_2} - \frac{28x_{\mathbb{C}_1 C_2}\mathbb{R}_1}{3} - \frac{14\mathbb{R}_1^2}{9}x_{\mathbb{C}_1 C_2}$$

$$= -2\sqrt{9 - \left(x_{\mathbb{C}_1 C_2} - 7\right)^2}\left(1 + \frac{\mathbb{R}_1}{3}\right)$$

$$4\left(1 + \frac{\mathbb{R}_1}{2}\right)^2 + 40\left(1 + \frac{\mathbb{R}_1}{3}\right)^2 - 1 - \frac{49\mathbb{R}_1^2}{9} - \frac{14x_{\mathbb{C}_1 C_2}\mathbb{R}_1}{3} - 14x_{\mathbb{C}_1 C_2} = -2\sqrt{9 - \left(x_{\mathbb{C}_1 C_2} - 7\right)^2}\left(1 + \frac{\mathbb{R}_1}{3}\right)$$

$$4\left(1 + \mathbb{R}_1 + \frac{\mathbb{R}_1^2}{4}\right) + 40\left(1 + \frac{2\mathbb{R}_1}{3} + \frac{\mathbb{R}_1^2}{9}\right) - 1 - \frac{49\mathbb{R}_1^2}{9} - 14x_{\mathbb{C}_1 C_2}\left(1 + \frac{\mathbb{R}_1}{3}\right) = -2\sqrt{9 - \left(x_{\mathbb{C}_1 C_2} - 7\right)^2}\left(1 + \frac{\mathbb{R}_1}{3}\right)$$

$$4 + 4\mathbb{R}_1 + \mathbb{R}_1^2 + 40 + \frac{80\mathbb{R}_1}{3} + \frac{40\mathbb{R}_1^2}{9} - 1 - \frac{49\mathbb{R}_1^2}{9} - 14x_{\mathbb{C}_1 C_2}\left(1 + \frac{\mathbb{R}_1}{3}\right) = -2\sqrt{9 - \left(x_{\mathbb{C}_1 C_2} - 7\right)^2}\left(1 + \frac{\mathbb{R}_1}{3}\right)$$

$$\left(\frac{92\mathbb{R}_1}{3} + 43\right) - 14x_{\mathbb{C}_1 C_2}\left(1 + \frac{\mathbb{R}_1}{3}\right) = -2\sqrt{9 - \left(x_{\mathbb{C}_1 C_2} - 7\right)^2}\left(1 + \frac{\mathbb{R}_1}{3}\right) \qquad \textbf{(8)}$$

Squaring the equation 8:

$$\left[\left(\frac{92\mathbb{R}_1}{3} + 43\right) - 14x_{\mathbb{C}_1 C_2}\left(1 + \frac{\mathbb{R}_1}{3}\right)\right]^2 = \left[-2\sqrt{9 - \left(x_{\mathbb{C}_1 C_2} - 7\right)^2}\left(1 + \frac{\mathbb{R}_1}{3}\right)\right]^2$$

$$\left[\left(\frac{92\mathbb{R}_1}{3} + 43\right) - 14x_{\mathbb{C}_1 C_2}\left(1 + \frac{\mathbb{R}_1}{3}\right)\right]^2 = 4\left[9 - \left(x_{\mathbb{C}_1 C_2} - 7\right)^2\right]\left(1 + \frac{\mathbb{R}_1}{3}\right)^2$$

$$\left(\frac{92\mathbb{R}_1}{3} + 43\right)^2 - 28x_{\mathbb{C}_1 C_2}\left(\frac{92\mathbb{R}_1}{3} + 43\right)\left(1 + \frac{\mathbb{R}_1}{3}\right) + 196x_{\mathbb{C}_1 C_2}^2\left(1 + \frac{\mathbb{R}_1}{3}\right)^2 = 36\left(1 + \frac{\mathbb{R}_1}{3}\right)^2 - 4\left(x_{\mathbb{C}_1 C_2} - 7\right)^2\left(1 + \frac{\mathbb{R}_1}{3}\right)^2$$

$$\left(\frac{92\mathbb{R}_1}{3} + 43\right)^2 - 28x_{\mathbb{C}_1 C_2}\left(\frac{92\mathbb{R}_1}{3} + 43\right)\left(1 + \frac{\mathbb{R}_1}{3}\right) + 196x_{\mathbb{C}_1 C_2}^2\left(1 + \frac{\mathbb{R}_1}{3}\right)^2 = 36\left(1 + \frac{\mathbb{R}_1}{3}\right)^2$$

$$-4\left(x_{\mathbb{C}_1 C_2}^2 - 14x_{\mathbb{C}_1 C_2} + 49\right)\left(1 + \frac{\mathbb{R}_1}{3}\right)^2$$

$$\left(\frac{92\mathbb{R}_1}{3} + 43\right)^2 - 28x_{\mathbb{C}_1 C_2}\left(\frac{92\mathbb{R}_1}{3} + 43\right)\left(1 + \frac{\mathbb{R}_1}{3}\right) + 196x_{\mathbb{C}_1 C_2}^2\left(1 + \frac{\mathbb{R}_1}{3}\right)^2 = 36\left(1 + \frac{\mathbb{R}_1}{3}\right)^2$$

$$-4x_{\mathbb{C}_1 C_2}^2\left(1 + \frac{\mathbb{R}_1}{3}\right)^2 + 56x_{\mathbb{C}_1 C_2}\left(1 + \frac{\mathbb{R}_1}{3}\right)^2 - 196\left(1 + \frac{\mathbb{R}_1}{3}\right)^2$$

$$200x_{\mathbb{C}_1 C_2}^2\left(1 + \frac{\mathbb{R}_1}{3}\right)^2 - 28x_{\mathbb{C}_1 C_2}\left(1 + \frac{\mathbb{R}_1}{3}\right)\left[\left(\frac{92\mathbb{R}_1}{3} + 43\right) + 2\left(1 + \frac{\mathbb{R}_1}{3}\right)\right] + \left(\frac{92\mathbb{R}_1}{3} + 43\right)^2 - 36\left(1 + \frac{\mathbb{R}_1}{3}\right)^2$$

$$+196\left(1 + \frac{\mathbb{R}_1}{3}\right)^2 = 0$$

$$200x_{\mathbb{C}_1 C_2}^2\left(1 + \frac{\mathbb{R}_1}{3}\right)^2 - 28x_{\mathbb{C}_1 C_2}\left(1 + \frac{\mathbb{R}_1}{3}\right)\left[\frac{92\mathbb{R}_1}{3} + 43 + 2 + \frac{2\mathbb{R}_1}{3}\right] + \left(\frac{92\mathbb{R}_1}{3} + 43\right)^2 + 160\left(1 + \frac{\mathbb{R}_1}{3}\right)^2 = 0$$

$$200x_{\mathbb{C}_1 C_2}^2\left(1 + \frac{\mathbb{R}_1}{3}\right)^2 - 28x_{\mathbb{C}_1 C_2}\left(1 + \frac{\mathbb{R}_1}{3}\right)\left(\frac{94\mathbb{R}_1}{3} + 45\right) + \frac{8464\mathbb{R}_1^2}{9} + \frac{7912\mathbb{R}_1}{3} + 1849 + 160\left(1 + \frac{2\mathbb{R}_1}{3} + \frac{\mathbb{R}_1^2}{9}\right) = 0$$

$$200x_{\mathbb{C}_1 C_2}^2\left(1 + \frac{\mathbb{R}_1}{3}\right)^2 - 28x_{\mathbb{C}_1 C_2}\left(1 + \frac{\mathbb{R}_1}{3}\right)\left(\frac{94\mathbb{R}_1}{3} + 45\right) + \frac{8464\mathbb{R}_1^2}{9} + \frac{7912\mathbb{R}_1}{3} + 1849 + 160 + \frac{320\mathbb{R}_1}{3} + \frac{160\mathbb{R}_1^2}{9} = 0$$

$$200x_{\mathbb{C}_1C_2}^2\left(1+\frac{\mathbb{R}_1}{3}\right)^2-28x_{\mathbb{C}_1C_2}\left(1+\frac{\mathbb{R}_1}{3}\right)\left(\frac{94\mathbb{R}_1}{3}+45\right)+\frac{8624\mathbb{R}_1^2}{9}+\frac{8232\mathbb{R}_1}{3}+2009=0$$

$$200x_{\mathbb{C}_1C_2}^2\left(1+\frac{\mathbb{R}_1}{3}\right)^2-28x_{\mathbb{C}_1C_2}\left(1+\frac{\mathbb{R}_1}{3}\right)\left(\frac{94\mathbb{R}_1}{3}+45\right)+\frac{8624\mathbb{R}_1^2}{9}+2744\mathbb{R}_1+2009=0$$

$$200\left(1+\frac{\mathbb{R}_1}{3}\right)^2x_{\mathbb{C}_1C_2}^2-28\left(1+\frac{\mathbb{R}_1}{3}\right)\left(\frac{94\mathbb{R}_1}{3}+45\right)x_{\mathbb{C}_1C_2}+\left(\frac{8624\mathbb{R}_1^2}{9}+2744\mathbb{R}_1+2009\right)=0 \quad \textbf{(9)}$$

From the equation 2:

$$x_{\mathbb{C}_1C_2}\left(1+\frac{\mathbb{R}_1}{3}\right)-\frac{7\mathbb{R}_1}{3}=x_{\mathbb{C}_1C_3}\left(1+\frac{\mathbb{R}_1}{4}\right)-\frac{3\mathbb{R}_1}{4}$$

$$x_{\mathbb{C}_1C_2}\left(1+\frac{\mathbb{R}_1}{3}\right)-\frac{7\mathbb{R}_1}{3}+\frac{3\mathbb{R}_1}{4}=x_{\mathbb{C}_1C_3}\left(1+\frac{\mathbb{R}_1}{4}\right)$$

$$x_{\mathbb{C}_1C_2}\left(1+\frac{\mathbb{R}_1}{3}\right)-\frac{28\mathbb{R}_1-9\mathbb{R}_1}{12}=x_{\mathbb{C}_1C_3}\left(1+\frac{\mathbb{R}_1}{4}\right)$$

$$x_{\mathbb{C}_1C_2}\left(1+\frac{\mathbb{R}_1}{3}\right)-\frac{19\mathbb{R}_1}{12}=x_{\mathbb{C}_1C_3}\left(1+\frac{\mathbb{R}_1}{4}\right)$$

$$x_{\mathbb{C}_1C_3}=\frac{x_{\mathbb{C}_1C_2}\left(1+\frac{\mathbb{R}_1}{3}\right)-\frac{19\mathbb{R}_1}{12}}{\left(1+\frac{\mathbb{R}_1}{4}\right)} \quad \textbf{(10)}$$

From the equation 4:

$$\sqrt{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2}\left(1+\frac{\mathbb{R}_1}{3}\right)-11=-\sqrt{16-\left(x_{\mathbb{C}_1C_3}-3\right)^2}\left(1+\frac{\mathbb{R}_1}{4}\right) \quad \textbf{(11)}$$

Squaring the equation 11:

$$\left[\sqrt{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2}\left(1+\frac{\mathbb{R}_1}{3}\right)-11\right]^2=\left[-\sqrt{16-\left(x_{\mathbb{C}_1C_3}-3\right)^2}\left(1+\frac{\mathbb{R}_1}{4}\right)\right]^2 \quad \textbf{(12)}$$

$$\left[9-\left(x_{\mathbb{C}_1C_2}-7\right)^2\right]\left(1+\frac{\mathbb{R}_1}{3}\right)^2-22\sqrt{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2}\left(1+\frac{\mathbb{R}_1}{3}\right)+121$$

$$=\left[16-\left(x_{\mathbb{C}_1C_3}-3\right)^2\right]\left(1+\frac{\mathbb{R}_1}{4}\right)^2$$

$$9\left(1+\frac{\mathbb{R}_1}{3}\right)^2-\left(x_{\mathbb{C}_1C_2}-7\right)^2\left(1+\frac{\mathbb{R}_1}{3}\right)^2-22\sqrt{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2}\left(1+\frac{\mathbb{R}_1}{3}\right)+121$$

$$=16\left(1+\frac{\mathbb{R}_1}{4}\right)^2-\left(x_{\mathbb{C}_1C_3}-3\right)^2\left(1+\frac{\mathbb{R}_1}{4}\right)^2$$

$$9\left(1+\frac{\mathbb{R}_1}{3}\right)^2-\left(x_{\mathbb{C}_1C_2}^2-14x_{\mathbb{C}_1C_2}+49\right)\left(1+\frac{\mathbb{R}_1}{3}\right)^2-22\sqrt{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2}\left(1+\frac{\mathbb{R}_1}{3}\right)$$

$$+121=16\left(1+\frac{\mathbb{R}_1}{4}\right)^2-\left(x_{\mathbb{C}_1C_3}^2-6x_{\mathbb{C}_1C_3}+9\right)\left(1+\frac{\mathbb{R}_1}{4}\right)^2$$

$$9\left(1+\frac{\mathbb{R}_1}{3}\right)^2-x_{\mathbb{C}_1C_2}^2\left(1+\frac{\mathbb{R}_1}{3}\right)^2+14x_{\mathbb{C}_1C_2}\left(1+\frac{\mathbb{R}_1}{3}\right)^2-49\left(1+\frac{\mathbb{R}_1}{3}\right)^2-22\sqrt{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2}\left(1+\frac{\mathbb{R}_1}{3}\right)$$

$$+121=16\left(1+\frac{\mathbb{R}_1}{4}\right)^2-x_{\mathbb{C}_1C_3}^2\left(1+\frac{\mathbb{R}_1}{4}\right)^2+6x_{\mathbb{C}_1C_3}\left(1+\frac{\mathbb{R}_1}{4}\right)^2-9\left(1+\frac{\mathbb{R}_1}{4}\right)^2$$

$$-40\left(1+\frac{\mathbb{R}_1}{3}\right)^2-x_{\mathbb{C}_1C_2}^2\left(1+\frac{\mathbb{R}_1}{3}\right)^2+14x_{\mathbb{C}_1C_2}\left(1+\frac{\mathbb{R}_1}{3}\right)^2-22\sqrt{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2}\left(1+\frac{\mathbb{R}_1}{3}\right)$$

$$+121=7\left(1+\frac{\mathbb{R}_1}{4}\right)^2-x_{\mathbb{C}_1C_3}^2\left(1+\frac{\mathbb{R}_1}{4}\right)^2+6x_{\mathbb{C}_1C_3}\left(1+\frac{\mathbb{R}_1}{4}\right)^2 \quad \textbf{(13)}$$

Replacing the equation 10 in the equation 13:

$$-40\left(1+\frac{\mathbb{R}_1}{3}\right)^2-x_{\mathbb{C}_1C_2}^2\left(1+\frac{\mathbb{R}_1}{3}\right)^2+14x_{\mathbb{C}_1C_2}\left(1+\frac{\mathbb{R}_1}{3}\right)^2-22\sqrt{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2}\left(1+\frac{\mathbb{R}_1}{3}\right)$$

$$+121=7\left(1+\frac{\mathbb{R}_1}{4}\right)^2-\left[\frac{x_{\mathbb{C}_1C_2}\left(1+\frac{\mathbb{R}_1}{3}\right)-\frac{19\mathbb{R}_1}{12}}{\left(1+\frac{\mathbb{R}_1}{4}\right)}\right]^2\left(1+\frac{\mathbb{R}_1}{4}\right)^2$$

$$+6\left[\frac{x_{\mathbb{C}_1C_2}\left(1+\frac{\mathbb{R}_1}{3}\right)-\frac{19\mathbb{R}_1}{12}}{\left(1+\frac{\mathbb{R}_1}{4}\right)}\right]\left(1+\frac{\mathbb{R}_1}{4}\right)^2$$

$$-40\left(1+\frac{\mathbb{R}_1}{3}\right)^2-x_{\mathbb{C}_1C_2}^2\left(1+\frac{\mathbb{R}_1}{3}\right)^2+14x_{\mathbb{C}_1C_2}\left(1+\frac{\mathbb{R}_1}{3}\right)^2-22\sqrt{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2}\left(1+\frac{\mathbb{R}_1}{3}\right)$$

$$+121=7\left(1+\frac{\mathbb{R}_1}{4}\right)^2-\left[\frac{x_{\mathbb{C}_1C_2}^2\left(1+\frac{\mathbb{R}_1}{3}\right)^2}{\left(1+\frac{\mathbb{R}_1}{4}\right)^2}-\frac{x_{\mathbb{C}_1C_2}\left(1+\frac{\mathbb{R}_1}{3}\right)\frac{19\mathbb{R}_1}{6}}{\left(1+\frac{\mathbb{R}_1}{4}\right)^2}+\frac{\frac{361\mathbb{R}_1^2}{144}}{\left(1+\frac{\mathbb{R}_1}{4}\right)^2}\right]\left(1+\frac{\mathbb{R}_1}{4}\right)^2$$

$$+6x_{\mathbb{C}_1C_2}\left(1+\frac{\mathbb{R}_1}{3}\right)\left(1+\frac{\mathbb{R}_1}{4}\right)-\frac{19\mathbb{R}_1}{2}\left(1+\frac{\mathbb{R}_1}{4}\right)$$

$$-40\left(1+\frac{\mathbb{R}_1}{3}\right)^2-x_{\mathbb{C}_1C_2}^2\left(1+\frac{\mathbb{R}_1}{3}\right)^2+14x_{\mathbb{C}_1C_2}\left(1+\frac{\mathbb{R}_1}{3}\right)^2-22\sqrt{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2}\left(1+\frac{\mathbb{R}_1}{3}\right)$$

$$+121=7\left(1+\frac{\mathbb{R}_1}{4}\right)^2-x_{\mathbb{C}_1C_2}^2\left(1+\frac{\mathbb{R}_1}{3}\right)^2+x_{\mathbb{C}_1C_2}\left(1+\frac{\mathbb{R}_1}{3}\right)\frac{19\mathbb{R}_1}{6}-\frac{361\mathbb{R}_1^2}{144}$$

$$+6x_{\mathbb{C}_1C_2}\left(1+\frac{\mathbb{R}_1}{3}\right)\left(1+\frac{\mathbb{R}_1}{4}\right)-\frac{19\mathbb{R}_1}{2}\left(1+\frac{\mathbb{R}_1}{4}\right)$$

$$121-40\left(1+\frac{\mathbb{R}_1}{3}\right)^2-7\left(1+\frac{\mathbb{R}_1}{4}\right)^2+\frac{361\mathbb{R}_1^2}{144}+\frac{19\mathbb{R}_1}{2}\left(1+\frac{\mathbb{R}_1}{4}\right)+14x_{\mathbb{C}_1C_2}\left(1+\frac{\mathbb{R}_1}{3}\right)^2$$

$$-x_{\mathbb{C}_1C_2}\left(1+\frac{\mathbb{R}_1}{3}\right)\frac{19\mathbb{R}_1}{6}-6x_{\mathbb{C}_1C_2}\left(1+\frac{\mathbb{R}_1}{3}\right)\left(1+\frac{\mathbb{R}_1}{4}\right)=22\sqrt{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2}\left(1+\frac{\mathbb{R}_1}{3}\right)$$

$$121-40\left(1+\frac{2\mathbb{R}_1}{3}+\frac{\mathbb{R}_1^2}{9}\right)-7\left(1+\frac{\mathbb{R}_1}{2}+\frac{\mathbb{R}_1^2}{16}\right)+\frac{361\mathbb{R}_1^2}{144}+\frac{19\mathbb{R}_1}{2}+\frac{19\mathbb{R}_1^2}{8}$$

$$x_{\mathbb{C}_1C_2}\left(1+\frac{\mathbb{R}_1}{3}\right)\left[14\left(1+\frac{\mathbb{R}_1}{3}\right)-\frac{19\mathbb{R}_1}{6}-6\left(1+\frac{\mathbb{R}_1}{4}\right)\right]=22\sqrt{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2}\left(1+\frac{\mathbb{R}_1}{3}\right)$$

$$121-40-\frac{80\mathbb{R}_1}{3}-\frac{40\mathbb{R}_1^2}{9}-7-\frac{7\mathbb{R}_1}{2}-\frac{7\mathbb{R}_1^2}{16}+\frac{361\mathbb{R}_1^2}{144}+\frac{19\mathbb{R}_1}{2}+\frac{19\mathbb{R}_1^2}{8}$$

$$x_{\mathbb{C}_1C_2}\left(1+\frac{\mathbb{R}_1}{3}\right)\left[14+\frac{14\mathbb{R}_1}{3}-\frac{19\mathbb{R}_1}{6}-6-\frac{3\mathbb{R}_1}{2}\right]=22\sqrt{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2}\left(1+\frac{\mathbb{R}_1}{3}\right)$$

$$\mathbb{R}_1^2\left(\frac{361}{144}+\frac{19}{8}-\frac{40}{9}-\frac{7}{16}\right)+\mathbb{R}_1\left(\frac{19}{2}-\frac{80}{3}-\frac{7}{2}\right)+74$$

$$x_{\mathbb{C}_1C_2}\left(1+\frac{\mathbb{R}_1}{3}\right)\left[8+\mathbb{R}_1\left(\frac{14}{3}-\frac{19}{6}-\frac{3}{2}\right)\right]=22\sqrt{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2}\left(1+\frac{\mathbb{R}_1}{3}\right)$$

$$\mathbb{R}_1^2\left(\frac{361+342-640-63}{144}\right)+\mathbb{R}_1\left(\frac{57-160-21}{6}\right)+74$$

$$x_{\mathbb{C}_1C_2}\left(1+\frac{\mathbb{R}_1}{3}\right)\left[8+\mathbb{R}_1\left(\frac{28-19-9}{6}\right)\right]=22\sqrt{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2}\left(1+\frac{\mathbb{R}_1}{3}\right)$$

$$\left(74-\frac{62\mathbb{R}_1}{3}\right)+8x_{\mathbb{C}_1C_2}\left(1+\frac{\mathbb{R}_1}{3}\right)=22\sqrt{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2}\left(1+\frac{\mathbb{R}_1}{3}\right) \qquad \textbf{(14)}$$

Squaring the equation 14:

$$\left[\left(74-\frac{62\mathbb{R}_1}{3}\right)+8x_{\mathbb{C}_1C_2}\left(1+\frac{\mathbb{R}_1}{3}\right)\right]^2=\left[22\sqrt{9-\left(x_{\mathbb{C}_1C_2}-7\right)^2}\left(1+\frac{\mathbb{R}_1}{3}\right)\right]^2$$

$$\left(74-\frac{62\mathbb{R}_1}{3}\right)^2+16x_{\mathbb{C}_1C_2}\left(1+\frac{\mathbb{R}_1}{3}\right)\left(74-\frac{62\mathbb{R}_1}{3}\right)+64x_{\mathbb{C}_1C_2}^2\left(1+\frac{\mathbb{R}_1}{3}\right)^2$$

$$=484\left[9-\left(x_{\mathbb{C}_1C_2}-7\right)^2\right]\left(1+\frac{\mathbb{R}_1}{3}\right)^2$$

$$\left(74-\frac{62\mathbb{R}_1}{3}\right)^2+16x_{\mathbb{C}_1C_2}\left(1+\frac{\mathbb{R}_1}{3}\right)\left(74-\frac{62\mathbb{R}_1}{3}\right)+64x_{\mathbb{C}_1C_2}^2\left(1+\frac{\mathbb{R}_1}{3}\right)^2$$

$$=4356\left(1+\frac{\mathbb{R}_1}{3}\right)^2-484\left(x_{\mathbb{C}_1C_2}-7\right)^2\left(1+\frac{\mathbb{R}_1}{3}\right)^2$$

$$\left(74-\frac{62\mathbb{R}_1}{3}\right)^2+16x_{\mathbb{C}_1C_2}\left(1+\frac{\mathbb{R}_1}{3}\right)\left(74-\frac{62\mathbb{R}_1}{3}\right)+64x_{\mathbb{C}_1C_2}^2\left(1+\frac{\mathbb{R}_1}{3}\right)^2$$

$$=4356\left(1+\frac{\mathbb{R}_1}{3}\right)^2-484\left(x_{\mathbb{C}_1C_2}^2-14x_{\mathbb{C}_1C_2}+49\right)\left(1+\frac{\mathbb{R}_1}{3}\right)^2$$

$$\left(74-\frac{62\mathbb{R}_1}{3}\right)^2+16x_{\mathbb{C}_1C_2}\left(1+\frac{\mathbb{R}_1}{3}\right)\left(74-\frac{62\mathbb{R}_1}{3}\right)+64x_{\mathbb{C}_1C_2}^2\left(1+\frac{\mathbb{R}_1}{3}\right)^2$$

$$=4356\left(1+\frac{\mathbb{R}_1}{3}\right)^2-484x_{\mathbb{C}_1C_2}^2\left(1+\frac{\mathbb{R}_1}{3}\right)^2+6776x_{\mathbb{C}_1C_2}\left(1+\frac{\mathbb{R}_1}{3}\right)^2-23716\left(1+\frac{\mathbb{R}_1}{3}\right)^2$$

$$548x_{\mathbb{C}_1C_2}^2\left(1+\frac{\mathbb{R}_1}{3}\right)^2+8x_{\mathbb{C}_1C_2}\left(1+\frac{\mathbb{R}_1}{3}\right)\left[2\left(74-\frac{62\mathbb{R}_1}{3}\right)-847\left(1+\frac{\mathbb{R}_1}{3}\right)\right]$$

$$+\left(74-\frac{62\mathbb{R}_1}{3}\right)^2+19360\left(1+\frac{\mathbb{R}_1}{3}\right)^2=0$$

$$548x_{\mathbb{C}_1C_2}^2\left(1+\frac{\mathbb{R}_1}{3}\right)^2+8x_{\mathbb{C}_1C_2}\left(1+\frac{\mathbb{R}_1}{3}\right)\left[148-\frac{124\mathbb{R}_1}{3}-847-\frac{847\mathbb{R}_1}{3}\right]$$

$$+5476-\frac{9176\mathbb{R}_1}{3}+\frac{3844\mathbb{R}_1^2}{9}+19360\left(1+\frac{2\mathbb{R}_1}{3}+\frac{\mathbb{R}_1^2}{9}\right)=0$$

$$548x_{\mathbb{C}_1C_2}^2\left(1+\frac{\mathbb{R}_1}{3}\right)^2+8x_{\mathbb{C}_1C_2}\left(1+\frac{\mathbb{R}_1}{3}\right)\left[-699-\frac{971\mathbb{R}_1}{3}\right]$$

$$+5476-\frac{9176\mathbb{R}_1}{3}+\frac{3844\mathbb{R}_1^2}{9}+19360+\frac{38720\mathbb{R}_1}{3}+\frac{19360\mathbb{R}_1^2}{9}=0$$

$$548x_{\mathbb{C}_1C_2}^2\left(1+\frac{\mathbb{R}_1}{3}\right)^2-8x_{\mathbb{C}_1C_2}\left(1+\frac{\mathbb{R}_1}{3}\right)\left[699+\frac{971\mathbb{R}_1}{3}\right]$$

$$+\frac{23204\mathbb{R}_1^2}{9}+\frac{29544\mathbb{R}_1}{3}+24836=0$$

$$548\left(1+\frac{\mathbb{R}_1}{3}\right)^2x_{\mathbb{C}_1C_2}^2-8\left(1+\frac{\mathbb{R}_1}{3}\right)\left[699+\frac{971\mathbb{R}_1}{3}\right]x_{\mathbb{C}_1C_2}$$

$$+\left(\frac{23204\mathbb{R}_1^2}{9}+9848\mathbb{R}_1+24836\right)=0 \qquad \textbf{(15)}$$

Solving the equation 9:

$$200\left(1+\frac{\mathbb{R}_1}{3}\right)^2x_{\mathbb{C}_1C_2}^2-28\left(1+\frac{\mathbb{R}_1}{3}\right)\left(\frac{94\mathbb{R}_1}{3}+45\right)x_{\mathbb{C}_1C_2}+\left(\frac{8624\mathbb{R}_1^2}{9}+2744\mathbb{R}_1+2009\right)=0$$

$$x_{\mathbb{C}_1C_2}=\frac{28\left(1+\frac{\mathbb{R}_1}{3}\right)\left(\frac{94\mathbb{R}_1}{3}+45\right)\pm\sqrt{784\left(1+\frac{\mathbb{R}_1}{3}\right)^2\left(\frac{94\mathbb{R}_1}{3}+45\right)^2-800\left(1+\frac{\mathbb{R}_1}{3}\right)^2\left(\frac{8624\mathbb{R}_1^2}{9}+2744\mathbb{R}_1+2009\right)}}{400\left(1+\frac{\mathbb{R}_1}{3}\right)^2}$$

$$x_{\mathbb{C}_1C_2}=\frac{28\left(1+\frac{\mathbb{R}_1}{3}\right)\left(\frac{94\mathbb{R}_1}{3}+45\right)\pm\left(1+\frac{\mathbb{R}_1}{3}\right)\sqrt{784\left(\frac{94\mathbb{R}_1}{3}+45\right)^2-800\left(\frac{8624\mathbb{R}_1^2}{9}+2744\mathbb{R}_1+2009\right)}}{400\left(1+\frac{\mathbb{R}_1}{3}\right)^2}$$

$$x_{\mathbb{C}_1 C_2} = \frac{28\left(\frac{94\mathbb{R}_1}{3} + 45\right) \pm \sqrt{784\left(\frac{8836\mathbb{R}_1^2}{9} + 2820\mathbb{R}_1 + 2025\right) - 800\left(\frac{8624\mathbb{R}_1^2}{9} + 2744\mathbb{R}_1 + 2009\right)}}{400\left(1 + \frac{\mathbb{R}_1}{3}\right)}$$

$$x_{\mathbb{C}_1 C_2} = \frac{28\left(\frac{94\mathbb{R}_1}{3} + 45\right) \pm \sqrt{\frac{6927424\mathbb{R}_1^2}{9} + 2210880\mathbb{R}_1 + 1587600 - \frac{6899200\mathbb{R}_1^2}{9} - 2195200\mathbb{R}_1 - 1607200}}{400\left(1 + \frac{\mathbb{R}_1}{3}\right)}$$

$$x_{\mathbb{C}_1 C_2} = \frac{28\left(\frac{94\mathbb{R}_1}{3} + 45\right) \pm \sqrt{3136\mathbb{R}_1^2 + 15680\mathbb{R}_1 - 19600}}{400\left(1 + \frac{\mathbb{R}_1}{3}\right)} \qquad \textbf{(16)}$$

There are two solutions from equation 16:

$$x_{\mathbb{C}_1 C_2 +} = \frac{28\left(\frac{94\mathbb{R}_1}{3} + 45\right) + \sqrt{3136\mathbb{R}_1^2 + 15680\mathbb{R}_1 - 19600}}{400\left(1 + \frac{\mathbb{R}_1}{3}\right)} \qquad \textbf{(17)}$$

$$x_{\mathbb{C}_1 C_2 -} = \frac{28\left(\frac{94\mathbb{R}_1}{3} + 45\right) - \sqrt{3136\mathbb{R}_1^2 + 15680\mathbb{R}_1 - 19600}}{400\left(1 + \frac{\mathbb{R}_1}{3}\right)} \qquad \textbf{(18)}$$

Replacing the equation 17 in the equation 15:

$$548\left(1 + \frac{\mathbb{R}_1}{3}\right)^2 \left[\frac{28\left(\frac{94\mathbb{R}_1}{3} + 45\right) + \sqrt{3136\mathbb{R}_1^2 + 15680\mathbb{R}_1 - 19600}}{400\left(1 + \frac{\mathbb{R}_1}{3}\right)}\right]^2$$

$$-8\left(1 + \frac{\mathbb{R}_1}{3}\right)\left[699 + \frac{971\mathbb{R}_1}{3}\right]\left[\frac{28\left(\frac{94\mathbb{R}_1}{3} + 45\right) + \sqrt{3136\mathbb{R}_1^2 + 15680\mathbb{R}_1 - 19600}}{400\left(1 + \frac{\mathbb{R}_1}{3}\right)}\right]$$

$$+\left(\frac{23204\mathbb{R}_1^2}{9} + 9848\mathbb{R}_1 + 24836\right) = 0$$

$$\frac{137}{40000}\left[784\left(\frac{94\mathbb{R}_1}{3} + 45\right)^2 + 56\left(\frac{94\mathbb{R}_1}{3} + 45\right)\sqrt{3136\mathbb{R}_1^2 + 15680\mathbb{R}_1 - 19600} + 3136\mathbb{R}_1^2 + 15680\mathbb{R}_1 - 19600\right]$$

$$-\frac{1}{50}\left[699 + \frac{971\mathbb{R}_1}{3}\right]\left[28\left(\frac{94\mathbb{R}_1}{3} + 45\right) + \sqrt{3136\mathbb{R}_1^2 + 15680\mathbb{R}_1 - 19600}\right]$$

$$+\left(\frac{23204\mathbb{R}_1^2}{9} + 9848\mathbb{R}_1 + 24836\right) = 0$$

$$\frac{137}{40000}\left[784\left(\frac{8836\mathbb{R}_1^2}{9} + \frac{8460\mathbb{R}_1}{3} + 2025\right) + 56\left(\frac{94\mathbb{R}_1}{3} + 45\right)\sqrt{3136\mathbb{R}_1^2 + 15680\mathbb{R}_1 - 19600} + 3136\mathbb{R}_1^2 + 15680\mathbb{R}_1 - 19600\right]$$

$$-\frac{14}{25}\left(699 + \frac{971\mathbb{R}_1}{3}\right)\left(\frac{94\mathbb{R}_1}{3} + 45\right) - \frac{1}{50}\left(699 + \frac{971\mathbb{R}_1}{3}\right)\sqrt{3136\mathbb{R}_1^2 + 15680\mathbb{R}_1 - 19600}$$

$$+\left(\frac{23204\mathbb{R}_1^2}{9} + 9848\mathbb{R}_1 + 24836\right) = 0$$

$$\frac{137}{40000}\left[\frac{6927424\mathbb{R}_1^2}{9}+2210880\mathbb{R}_1+1587600+56\left(\frac{94\mathbb{R}_1}{3}+45\right)\sqrt{3136\mathbb{R}_1^2+15680\mathbb{R}_1-19600}+3136\mathbb{R}_1^2+15680\mathbb{R}_1-19600\right]$$

$$-\frac{14}{25}\left(21902\mathbb{R}_1+31455+\frac{91274\mathbb{R}_1^2}{9}+14565\mathbb{R}_1\right)-\frac{1}{50}\left(699+\frac{971\mathbb{R}_1}{3}\right)\sqrt{3136\mathbb{R}_1^2+15680\mathbb{R}_1-19600}$$

$$+\left(\frac{23204\mathbb{R}_1^2}{9}+9848\mathbb{R}_1+24836\right)=0$$

$$\frac{137}{40000}\left[\frac{6955648\mathbb{R}_1^2}{9}+2226560\mathbb{R}_1+1568000+56\left(\frac{94\mathbb{R}_1}{3}+45\right)\sqrt{3136\mathbb{R}_1^2+15680\mathbb{R}_1-19600}\right]$$

$$-\frac{14}{25}\left(\frac{91274\mathbb{R}_1^2}{9}+36467\mathbb{R}_1+31455\right)-\frac{1}{50}\left(699+\frac{971\mathbb{R}_1}{3}\right)\sqrt{3136\mathbb{R}_1^2+15680\mathbb{R}_1-19600}$$

$$+\left(\frac{23204\mathbb{R}_1^2}{9}+9848\mathbb{R}_1+24836\right)=0$$

$$\frac{14889434\mathbb{R}_1^2}{5625}+\frac{953246\mathbb{R}_1}{125}+\frac{26852}{5}+\frac{959}{5000}\left(\frac{94\mathbb{R}_1}{3}+45\right)\sqrt{3136\mathbb{R}_1^2+15680\mathbb{R}_1-19600}$$

$$-\frac{1277836\mathbb{R}_1^2}{225}-\frac{510538\mathbb{R}_1}{25}-\frac{88074}{5}-\frac{1}{50}\left(699+\frac{971\mathbb{R}_1}{3}\right)\sqrt{3136\mathbb{R}_1^2+15680\mathbb{R}_1-19600}$$
$$+\left(\frac{23204\mathbb{R}_1^2}{9}+9848\mathbb{R}_1+24836\right)=0$$

$$\frac{14889434-31945900+14502500}{5625}\mathbb{R}_1^2+\frac{953246-2552690+1231000}{125}\mathbb{R}_1+\frac{26852-88074+124180}{5}$$

$$+\sqrt{3136\mathbb{R}_1^2+15680\mathbb{R}_1-19600}\left[\frac{959}{5000}\left(\frac{94\mathbb{R}_1}{3}+45\right)-\frac{1}{50}\left(699+\frac{971\mathbb{R}_1}{3}\right)\right]=0$$

$$-\frac{2553966}{5625}\mathbb{R}_1^2-\frac{368444}{125}\mathbb{R}_1+\frac{62958}{5}=-\sqrt{3136\mathbb{R}_1^2+15680\mathbb{R}_1-19600}\left[\frac{45073\mathbb{R}_1}{7500}+\frac{8631}{1000}-\frac{699}{50}-\frac{971\mathbb{R}_1}{150}\right]$$

$$-\frac{2553966}{5625}\mathbb{R}_1^2-\frac{368444}{125}\mathbb{R}_1+\frac{62958}{5}=-\sqrt{3136\mathbb{R}_1^2+15680\mathbb{R}_1-19600}\left[\frac{45073-48550}{7500}\mathbb{R}_1+\frac{8631-13980}{1000}\right]$$

$$-\left(\frac{2553966}{5625}\mathbb{R}_1^2+\frac{368444}{125}\mathbb{R}_1-\frac{62958}{5}\right)=-\sqrt{3136\mathbb{R}_1^2+15680\mathbb{R}_1-19600}\left(-\frac{3477}{7500}\mathbb{R}_1-\frac{5349}{1000}\right)$$

$$-\left(\frac{2553966}{5625}\mathbb{R}_1^2+\frac{368444}{125}\mathbb{R}_1-\frac{62958}{5}\right)=\sqrt{3136\mathbb{R}_1^2+15680\mathbb{R}_1-19600}\left(\frac{3477}{7500}\mathbb{R}_1+\frac{5349}{1000}\right)\qquad\textbf{(19)}$$

Squaring the equation 19:

$$\left[-\left(\frac{2553966}{5625}\mathbb{R}_1^2+\frac{368444}{125}\mathbb{R}_1-\frac{62958}{5}\right)\right]^2=\left[\sqrt{3136\mathbb{R}_1^2+15680\mathbb{R}_1-19600}\left(\frac{3477}{7500}\mathbb{R}_1+\frac{5349}{1000}\right)\right]^2$$

$$\frac{(2553966)^2}{(5625)^2}\mathbb{R}_1^4+\frac{2553966(368444)}{5625(125)}\mathbb{R}_1^3-\frac{2553966(62958)}{5625(5)}\mathbb{R}_1^2+\frac{368444(2553966)}{125(5625)}\mathbb{R}_1^3+\frac{(368444)^2}{(125)^2}\mathbb{R}_1^2$$

$$-\frac{368444(62958)}{125(5)}\mathbb{R}_1-\frac{62958(2553966)}{5(5625)}\mathbb{R}_1^2-\frac{62958(368444)}{5(125)}\mathbb{R}_1+\frac{(62958)^2}{(5)^2}=$$

$$(3136\mathbb{R}_1^2+15680\mathbb{R}_1-19600)\left(\frac{(3477)^2}{(7500)^2}\mathbb{R}_1^2+\frac{3477(5349)}{7500(500)}\mathbb{R}_1+\frac{(5349)^2}{(1000)^2}\right)$$

$$\frac{(2553966)^2}{(5625)^2}\mathbb{R}_1^4+\frac{2553966(368444)}{5625(125)}\mathbb{R}_1^3-\frac{2553966(62958)}{5625(5)}\mathbb{R}_1^2+\frac{368444(2553966)}{125(5625)}\mathbb{R}_1^3+\frac{(368444)^2}{(125)^2}\mathbb{R}_1^2$$

$$-\frac{368444(62958)}{125(5)}\mathbb{R}_1-\frac{62958(2553966)}{5(5625)}\mathbb{R}_1^2-\frac{62958(368444)}{5(125)}\mathbb{R}_1+\frac{(62958)^2}{(5)^2}=$$

$$\frac{3136(3477)^2}{(7500)^2}\mathbb{R}_1^4+\frac{3136(3477)(5349)}{7500(500)}\mathbb{R}_1^3+\frac{3136(5349)^2}{(1000)^2}\mathbb{R}_1^2+\frac{15680(3477)^2}{(7500)^2}\mathbb{R}_1^3$$

$$+\frac{15680(3477)(5349)}{7500(500)}\mathbb{R}_1^2+\frac{15680(5349)^2}{(1000)^2}\mathbb{R}_1-\frac{19600(3477)^2}{(7500)^2}\mathbb{R}_1^2-\frac{19600(3477)(5349)}{7500(500)}\mathbb{R}_1$$

$$-\frac{19600(5349)^2}{(1000)^2}$$

$$\left[\frac{(2553966)^2}{(5625)^2}-\frac{3136(3477)^2}{(7500)^2}\right]\mathbb{R}_1^4$$

$$+\left[\frac{2553966(368444)}{5625(125)}+\frac{368444(2553966)}{125(5625)}-\frac{3136(3477)(5349)}{7500(500)}-\frac{15680(3477)^2}{(7500)^2}\right]\mathbb{R}_1^3$$

$$+\left[\frac{(368444)^2}{(125)^2}-\frac{2553966(62958)}{5625(5)}-\frac{62958(2553966)}{5(5625)}-\frac{3136(5349)^2}{(1000)^2}-\frac{15680(3477)(5349)}{7500(500)}+\frac{19600(3477)^2}{(7500)^2}\right]\mathbb{R}_1^2$$

$$+\left[\frac{19600(3477)(5349)}{7500(500)}-\frac{368444(62958)}{125(5)}-\frac{62958(368444)}{5(125)}-\frac{15680(5349)^2}{(1000)^2}\right]\mathbb{R}_1$$

$$+\frac{(62958)^2}{(5)^2}+\frac{19600(5349)^2}{(1000)^2}=0$$

$$\frac{25684608}{125}\mathbb{R}_1^4+\frac{1661050176}{625}\mathbb{R}_1^3-\frac{363669692}{125}\mathbb{R}_1^2$$

$$-\frac{46612635342}{625}\mathbb{R}_1+\frac{397772954649}{2500}=0$$

$\mathbb{R}_{11}\approx -9{,}48891$

$\mathbb{R}_{12}\approx -9{,}37282$

$\mathbb{R}_{13}\approx 2{,}68562$

$\mathbb{R}_{14}\approx 3{,}24191$

There are four solutions for $\mathbb{R}_1$. $\mathbb{R}_{11}$, $\mathbb{R}_{12}$ are discarded. $\mathbb{R}_{14}$ is also discarded, a tangent circle to the three given circles is not obtained with this solution. $\mathbb{R}_{13}$ is taken as the solution for $\mathbb{R}_{1+}$:

$$\mathbb{R}_{1+}\approx 2{,}68562 \qquad \textbf{(20)}$$

Replacing the equation 20 in the equation 17:

$$x_{\mathbb{C}_1C_2+}\approx\frac{28\left(\frac{94*2{,}68562}{3}+45\right)+\sqrt{3136(2{,}68562)^2+15680*2{,}68562-19600}}{400\left(1+\frac{2{,}68562}{3}\right)}$$

$$x_{\mathbb{C}_1 C_2 +} \approx 5{,}05040 \tag{21}$$

Replacing the equations 20 and 21 in the equation 10:

$$x_{\mathbb{C}_1 C_3 +} \approx \frac{5{,}05040\left(1+\frac{2{,}68562}{3}\right)-\frac{19*2{,}68562}{12}}{\left(1+\frac{2{,}68562}{4}\right)}$$

$$x_{\mathbb{C}_1 C_3 +} \approx 3{,}18254 \tag{22}$$

Replacing the equations 20 and 21 in the equation 5:

$$x_{\mathbb{C}_1 C_1 +} \approx \frac{5{,}05040+\frac{2{,}68562(5{,}05040-7)}{3}}{1+\frac{2{,}68562}{2}}$$

$$x_{\mathbb{C}_1 C_1 +} \approx 1{,}41074 \tag{23}$$

The equations 137, 142 and 147 (main text) have the format $\left(x-h_{\mathbb{C}_1}\right)^2+\left(y-k_{\mathbb{C}_1}\right)^2=\mathbb{R}_1^2$. $h_{\mathbb{C}_1}$ and $k_{\mathbb{C}_1}$ can be calculated from any of these equations. Taking the equation 137:

$$h_{\mathbb{C}_1 +} = x_{\mathbb{C}_1 C_1 +} + \frac{\mathbb{R}_{1+} x_{\mathbb{C}_1 C_1 +}}{2} \tag{24}$$

$$k_{\mathbb{C}_1 +} = \sqrt{4-x_{\mathbb{C}_1 C_1 +}^2}\left(1+\frac{\mathbb{R}_{1+}}{2}\right) \tag{25}$$

Replacing the equations 20 and 23 in the equation 24:

$$h_{\mathbb{C}_1 +} \approx 1{,}41074+\frac{2{,}68562*1{,}41074}{2}$$

$$h_{\mathbb{C}_1 +} \approx 3{,}30510 \tag{26}$$

Replacing the equations 20 and 23 in the equation 25:

$$k_{\mathbb{C}_1 +} \approx \sqrt{4-(1{,}41074)^2}\left(1+\frac{2{,}68562}{2}\right)$$

$$k_{\mathbb{C}_1 +} \approx 3{,}32134 \tag{27}$$

Replacing the equation 18 in the equation 15:

$$548\left(1+\frac{\mathbb{R}_1}{3}\right)^2\left[\frac{28\left(\frac{94\mathbb{R}_1}{3}+45\right)-\sqrt{3136\mathbb{R}_1^2+15680\mathbb{R}_1-19600}}{400\left(1+\frac{\mathbb{R}_1}{3}\right)}\right]^2$$

$$-8\left(1+\frac{\mathbb{R}_1}{3}\right)\left[699+\frac{971\mathbb{R}_1}{3}\right]\left[\frac{28\left(\frac{94\mathbb{R}_1}{3}+45\right)-\sqrt{3136\mathbb{R}_1^2+15680\mathbb{R}_1-19600}}{400\left(1+\frac{\mathbb{R}_1}{3}\right)}\right]$$

$$+\left(\frac{23204\mathbb{R}_1^2}{9}+9848\mathbb{R}_1+24836\right)=0$$

$$\frac{137}{40000}\left[784\left(\frac{94\mathbb{R}_1}{3}+45\right)^2-56\left(\frac{94\mathbb{R}_1}{3}+45\right)\sqrt{3136\mathbb{R}_1^2+15680\mathbb{R}_1-19600}+3136\mathbb{R}_1^2+15680\mathbb{R}_1-19600\right]$$

$$-\frac{1}{50}\left[699+\frac{971\mathbb{R}_1}{3}\right]\left[28\left(\frac{94\mathbb{R}_1}{3}+45\right)-\sqrt{3136\mathbb{R}_1^2+15680\mathbb{R}_1-19600}\right]$$

$$+\left(\frac{23204\mathbb{R}_1^2}{9}+9848\mathbb{R}_1+24836\right)=0$$

$$\frac{137}{40000}\left[784\left(\frac{8836\mathbb{R}_1^2}{9}+\frac{8460\mathbb{R}_1}{3}+2025\right)-56\left(\frac{94\mathbb{R}_1}{3}+45\right)\sqrt{3136\mathbb{R}_1^2+15680\mathbb{R}_1-19600}+3136\mathbb{R}_1^2+15680\mathbb{R}_1-19600\right]$$

$$-\frac{14}{25}\left(699+\frac{971\mathbb{R}_1}{3}\right)\left(\frac{94\mathbb{R}_1}{3}+45\right)+\frac{1}{50}\left(699+\frac{971\mathbb{R}_1}{3}\right)\sqrt{3136\mathbb{R}_1^2+15680\mathbb{R}_1-19600}$$

$$+\left(\frac{23204\mathbb{R}_1^2}{9}+9848\mathbb{R}_1+24836\right)=0$$

$$\frac{137}{40000}\left[\frac{6927424\mathbb{R}_1^2}{9}+2210880\mathbb{R}_1+1587600-56\left(\frac{94\mathbb{R}_1}{3}+45\right)\sqrt{3136\mathbb{R}_1^2+15680\mathbb{R}_1-19600}+3136\mathbb{R}_1^2+15680\mathbb{R}_1-19600\right]$$

$$-\frac{14}{25}\left(21902\mathbb{R}_1+31455+\frac{91274\mathbb{R}_1^2}{9}+14565\mathbb{R}_1\right)+\frac{1}{50}\left(699+\frac{971\mathbb{R}_1}{3}\right)\sqrt{3136\mathbb{R}_1^2+15680\mathbb{R}_1-19600}$$

$$+\left(\frac{23204\mathbb{R}_1^2}{9}+9848\mathbb{R}_1+24836\right)=0$$

$$\frac{137}{40000}\left[\frac{6955648\mathbb{R}_1^2}{9}+2226560\mathbb{R}_1+1568000-56\left(\frac{94\mathbb{R}_1}{3}+45\right)\sqrt{3136\mathbb{R}_1^2+15680\mathbb{R}_1-19600}\right]$$

$$-\frac{14}{25}\left(\frac{91274\mathbb{R}_1^2}{9}+36467\mathbb{R}_1+31455\right)+\frac{1}{50}\left(699+\frac{971\mathbb{R}_1}{3}\right)\sqrt{3136\mathbb{R}_1^2+15680\mathbb{R}_1-19600}$$

$$+\left(\frac{23204\mathbb{R}_1^2}{9}+9848\mathbb{R}_1+24836\right)=0$$

$$\frac{14889434\mathbb{R}_1^2}{5625}+\frac{953246\mathbb{R}_1}{125}+\frac{26852}{5}-\frac{959}{5000}\left(\frac{94\mathbb{R}_1}{3}+45\right)\sqrt{3136\mathbb{R}_1^2+15680\mathbb{R}_1-19600}$$

$$-\frac{1277836\mathbb{R}_1^2}{225}-\frac{510538\mathbb{R}_1}{25}-\frac{88074}{5}+\frac{1}{50}\left(699+\frac{971\mathbb{R}_1}{3}\right)\sqrt{3136\mathbb{R}_1^2+15680\mathbb{R}_1-19600}$$
$$+\left(\frac{23204\mathbb{R}_1^2}{9}+9848\mathbb{R}_1+24836\right)=0$$

$$\frac{14889434-31945900+14502500}{5625}\mathbb{R}_1^2+\frac{953246-2552690+1231000}{125}\mathbb{R}_1+\frac{26852-88074+124180}{5}$$

$$+\sqrt{3136\mathbb{R}_1^2+15680\mathbb{R}_1-19600}\left[\frac{1}{50}\left(699+\frac{971\mathbb{R}_1}{3}\right)-\frac{959}{5000}\left(\frac{94\mathbb{R}_1}{3}+45\right)\right]=0$$

$$-\frac{2553966}{5625}\mathbb{R}_1^2-\frac{368444}{125}\mathbb{R}_1+\frac{62958}{5}=-\sqrt{3136\mathbb{R}_1^2+15680\mathbb{R}_1-19600}\left[\frac{699}{50}+\frac{971\mathbb{R}_1}{150}-\frac{45073\mathbb{R}_1}{7500}-\frac{8631}{1000}\right]$$

$$-\frac{2553966}{5625}\mathbb{R}_1^2-\frac{368444}{125}\mathbb{R}_1+\frac{62958}{5}=-\sqrt{3136\mathbb{R}_1^2+15680\mathbb{R}_1-19600}\left[\frac{48550-45073}{7500}\mathbb{R}_1+\frac{13980-8631}{1000}\right]$$

$$-\left(\frac{2553966}{5625}\mathbb{R}_1^2+\frac{368444}{125}\mathbb{R}_1-\frac{62958}{5}\right)=-\sqrt{3136\mathbb{R}_1^2+15680\mathbb{R}_1-19600}\left(\frac{3477}{7500}\mathbb{R}_1+\frac{5349}{1000}\right) \quad \textbf{(28)}$$

Squaring the equation 28:

$$\left[-\left(\frac{2553966}{5625}\mathbb{R}_1^2+\frac{368444}{125}\mathbb{R}_1-\frac{62958}{5}\right)\right]^2=\left[-\sqrt{3136\mathbb{R}_1^2+15680\mathbb{R}_1-19600}\left(\frac{3477}{7500}\mathbb{R}_1+\frac{5349}{1000}\right)\right]^2$$

$$\frac{(2553966)^2}{(5625)^2}\mathbb{R}_1^4+\frac{2553966(368444)}{5625(125)}\mathbb{R}_1^3-\frac{2553966(62958)}{5625(5)}\mathbb{R}_1^2+\frac{368444(2553966)}{125(5625)}\mathbb{R}_1^3+\frac{(368444)^2}{(125)^2}\mathbb{R}_1^2$$

$$-\frac{368444(62958)}{125(5)}\mathbb{R}_1-\frac{62958(2553966)}{5(5625)}\mathbb{R}_1^2-\frac{62958(368444)}{5(125)}\mathbb{R}_1+\frac{(62958)^2}{(5)^2}=$$

$$(3136\mathbb{R}_1^2+15680\mathbb{R}_1-19600)\left(\frac{(3477)^2}{(7500)^2}\mathbb{R}_1^2+\frac{3477(5349)}{7500(500)}\mathbb{R}_1+\frac{(5349)^2}{(1000)^2}\right)$$

$$\frac{(2553966)^2}{(5625)^2}\mathbb{R}_1^4+\frac{2553966(368444)}{5625(125)}\mathbb{R}_1^3-\frac{2553966(62958)}{5625(5)}\mathbb{R}_1^2+\frac{368444(2553966)}{125(5625)}\mathbb{R}_1^3+\frac{(368444)^2}{(125)^2}\mathbb{R}_1^2$$

$$-\frac{368444(62958)}{125(5)}\mathbb{R}_1-\frac{62958(2553966)}{5(5625)}\mathbb{R}_1^2-\frac{62958(368444)}{5(125)}\mathbb{R}_1+\frac{(62958)^2}{(5)^2}=$$

$$\frac{3136(3477)^2}{(7500)^2}\mathbb{R}_1^4+\frac{3136(3477)(5349)}{7500(500)}\mathbb{R}_1^3+\frac{3136(5349)^2}{(1000)^2}\mathbb{R}_1^2+\frac{15680(3477)^2}{(7500)^2}\mathbb{R}_1^3$$

$$+\frac{15680(3477)(5349)}{7500(500)}\mathbb{R}_1^2+\frac{15680(5349)^2}{(1000)^2}\mathbb{R}_1-\frac{19600(3477)^2}{(7500)^2}\mathbb{R}_1^2-\frac{19600(3477)(5349)}{7500(500)}\mathbb{R}_1$$

$$-\frac{19600(5349)^2}{(1000)^2}$$

$$\left[\frac{(2553966)^2}{(5625)^2}-\frac{3136(3477)^2}{(7500)^2}\right]\mathbb{R}_1^4$$

$$+\left[\frac{2553966(368444)}{5625(125)}+\frac{368444(2553966)}{125(5625)}-\frac{3136(3477)(5349)}{7500(500)}-\frac{15680(3477)^2}{(7500)^2}\right]\mathbb{R}_1^3$$

$$+\left[\frac{(368444)^2}{(125)^2}-\frac{2553966(62958)}{5625(5)}-\frac{62958(2553966)}{5(5625)}-\frac{3136(5349)^2}{(1000)^2}-\frac{15680(3477)(5349)}{7500(500)}+\frac{19600(3477)^2}{(7500)^2}\right]\mathbb{R}_1^2$$

$$+\left[\frac{19600(3477)(5349)}{7500(500)}-\frac{368444(62958)}{125(5)}-\frac{62958(368444)}{5(125)}-\frac{15680(5349)^2}{(1000)^2}\right]\mathbb{R}_1$$

$$+\frac{(62958)^2}{(5)^2}+\frac{19600(5349)^2}{(1000)^2}=0$$

$$\frac{25684608}{125}\mathbb{R}_1^4+\frac{1661050176}{625}\mathbb{R}_1^3-\frac{363669692}{125}\mathbb{R}_1^2$$

$$-\frac{46612635342}{625}\mathbb{R}_1+\frac{397772954649}{2500}=0$$

$$\mathbb{R}_{11}\approx -9{,}48891$$

$$\mathbb{R}_{12}\approx -9{,}37282$$

$\mathbb{R}_{13} \approx 2{,}68562$

$\mathbb{R}_{14} \approx 3{,}24191$

There are four solutions for $\mathbb{R}_1$. $\mathbb{R}_{11}$, $\mathbb{R}_{12}$ are discarded. $\mathbb{R}_{14}$ is also discarded, a tangent circle to the three given circles is not obtained with this solution. $\mathbb{R}_{13}$ is taken as the solution for $\mathbb{R}_{1-}$:

$$\mathbb{R}_{1-} \approx 2{,}68562 \qquad \textbf{(29)}$$

Replacing the equations 17 or 18 in the equation 15 originates the same polynomial with a degree of four. Therefore, the solutions are the same. Taking $\mathbb{R}_{1+}$ or $\mathbb{R}_{1-}$ as the solution for $\mathbb{R}_1$.

$$\mathbb{R}_1 \approx 2{,}68562 \qquad \textbf{(30)}$$

Similarly:

$x_{\mathbb{C}_1 C_2} \approx 5{,}05040$

$x_{\mathbb{C}_1 C_3} \approx 3{,}18254$

$x_{\mathbb{C}_1 C_1} \approx 1{,}41074$

$h_{\mathbb{C}_1} \approx 3{,}30510$

$k_{\mathbb{C}_1} \approx 3{,}32134$

Appendix H. Derivatives at the tangency points for the tangent circle $\mathbb{C}_1$

$f(x) = y = +\sqrt{4 - x^2}$ (upper semicircle C_1)

$$f'(x) = y' = -\frac{x}{\sqrt{4 - x^2}}$$

$f(x) = y = +\sqrt{9 - (x-7)^2} - 1$ (upper semicircle C_2)

$$f'(x) = y' = -\frac{x - 7}{\sqrt{9 - (x-7)^2}}$$

$f(x) = y = -\sqrt{16 - (x-3)^2} + 10$ (lower semicircle C_3)

$$f'(x) = y' = \frac{x - 3}{\sqrt{16 - (x-3)^2}}$$

$x_{\mathbb{C}_1 C_1} \approx 1{,}41074$

$x_{\mathbb{C}_1 C_2} \approx 5{,}05040$

$x_{\mathbb{C}_1 C_3} \approx 3{,}18254$

$\mathbb{R}_1 \approx 2{,}68562$

$h_{\mathbb{C}_1} \approx 3{,}30510$

$k_{\mathbb{C}_1} \approx 3{,}32134$

$f'\left(x_{\mathbb{C}_1 C_1}\right) \approx -\frac{1{,}41074}{\sqrt{4-(1{,}41074)^2}} \approx -0{,}99511$ (upper semicircle C_1)

$f'(x_{\mathbb{C}_1 C_2}) \approx -\frac{5{,}05040-7}{\sqrt{9-(5{,}05040-7)^2}} \approx 0{,}85503$ (upper semicircle C_2)

$f'(x_{\mathbb{C}_1 C_3}) \approx \frac{3{,}18254-3}{\sqrt{16-(3{,}18254-3)^2}} \approx 0{,}04568$ (lower semicircle C_3)

The equation of a circle with center $\left(h_{\mathbb{C}_1}, k_{\mathbb{C}_1}\right)$ and radius $\mathbb{R}_1$ is given by:

$$\left(x - h_{\mathbb{C}_1}\right)^2 + \left(y - k_{\mathbb{C}_1}\right)^2 = \mathbb{R}_1^2$$

Where two functions can be obtained:

$$f(x) = \sqrt{\mathbb{R}_1^2 - \left(x - h_{\mathbb{C}_1}\right)^2} + k_{\mathbb{C}_1}$$

$$f'(x) = -\frac{x - h_{\mathbb{C}_1}}{\sqrt{\mathbb{R}_1^2 - \left(x - h_{\mathbb{C}_1}\right)^2}}$$

$$f(x) = -\sqrt{\mathbb{R}_1^2 - \left(x - h_{\mathbb{C}_1}\right)^2} + k_{\mathbb{C}_1}$$

$$f'(x) = \frac{x - h_{\mathbb{C}_1}}{\sqrt{\mathbb{R}_1^2 - \left(x - h_{\mathbb{C}_1}\right)^2}}$$

$f'\left(x_{\mathbb{C}_1 C_1}\right) \approx \frac{1{,}41074-3{,}30510}{\sqrt{(2{,}68562)^2-(1{,}41074-3{,}30510)^2}} \approx -0{,}99511$ derivative of the tangent circle $\mathbb{C}_1$ at the tangency point $x_{\mathbb{C}_1 C_1}$

$f'\left(x_{\mathbb{C}_1 C_2}\right) \approx \frac{5{,}05040-3{,}30510}{\sqrt{(2{,}68562)^2-(5{,}05040-3{,}30510)^2}} \approx 0{,}85503$ derivative of the tangent circle $\mathbb{C}_1$ at the tangency point $x_{\mathbb{C}_1 C_2}$

$f'\left(x_{\mathbb{C}_1 C_3}\right) \approx -\frac{3{,}18254-3{,}30510}{\sqrt{(2{,}68562)^2-(3{,}18254-3{,}30510)^2}} \approx 0{,}04568$ derivative of the tangent circle $\mathbb{C}_1$ at the tangency point $x_{\mathbb{C}_1 C_3}$

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