MATEMÁTICA
ESPOL - FCNM JOURNAL
2023, Vol. 21, No. 2 Julio - Diciembre

Frail and strong solutions for a *p*-Laplace boundary problem with infinitely many discontinuities

Soluciones frágiles y fuertes para un problema de valor en la frontera con operador *p*-Laplaciano y con infinitas discontinuidades

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Recepción: 13/11/2022 **Aceptación:** 07/06/2023 **Publicación:** 31/07/2023

Abstract We consider the problem (P) $-\Delta_p u(x) = h(x)\, f(u(x)) + q(x)$, $x \in \Omega$, with $u(x) = 0$, $x \in \partial\Omega$, where $p > 1$, $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with smooth boundary, $q \in \mathrm{L}^{p'}(\Omega)$, $1/p + 1/p' = 1$, $h \in \mathrm{L}^\infty(\Omega) \setminus \{0\}$. We assume that f has a countable set of upward and downward discontinuities, $D \subseteq \mathbb{R}$, and verifies $|f(s)| \leq C_1 + C_2 |s|^\alpha$, $s \in \mathbb{R}$, where $\alpha, C_1, C_2 > 0$ and $1 + \alpha \in [p, p^*]$, $p^* = pN/(N-p)$. Since the standard functional, I, associated to (P) is not Fréchet differentiable but locally Lipschitz continuous on $\mathrm{W}_0^{1,p}(\Omega)$, we apply the variational tools developed by Chang and Clarke. We characterize a *frail solution* of (P), one that verifies a.e. a condition involving an appropriate multivalued function, as a generalized critical point of I. Given u, a frail solution of (P), we find sufficient conditions for $u^{-1}(D)$ to have zero measure; this is enough for u to become a *strong solution* of (P): it satisfies (P) a.e. We show conditions for the existence of local-extremum strong solutions of (P). Finally we prove that if f verifies a growing condition involving the first eigenvalue of $-\Delta_p$, then (P) has a *ground state*, i.e., a strong solution which globaly minimizes I.
Keywords boundary value problem, frail solution, non-differentiable functional, p-Laplace operator, strong solution.

Resumen Consideramos el problema $-\Delta_p u(x) = h(x)\, f(u(x)) + q(x)$, $x \in \Omega$, con $u(x) = 0$, $x \in \partial\Omega$, donde $p > 1$, $\Omega \subseteq \mathbb{R}^N$ es un dominio acotado con frontera suave, $q \in \mathrm{L}^{p'}(\Omega)$, $1/p + 1/p' = 1$, $h \in \mathrm{L}^\infty(\Omega) \setminus \{0\}$. Suponemos que f tiene un con-

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junto contable de discontinuidades de salto, $D \subseteq \mathbb{R}$, y verifica $|f(s)| \leq C_1 + C_2 |s|^{\alpha}$, $s \in \mathbb{R}$, donde $\alpha, C_1, C_2 > 0$ y $1 + \alpha \in [p, p^*]$, $p^* = pN/(N-p)$. Puesto que I, el funcional asociado a (P) no es Fréchet diferenciable sino localmente Lipschitz continua sobre $\mathrm{W}_0^{1,p}(\Omega)$, aplicamos las herramientas variacionales desarrolladas por Chang y Clarke. Caracterizamos una *solución frágil* de (P), una que verifica c.t.p. una condición que involucra una adecuada función multivaluada, como un punto crítico generalizado de I. Dada u, una solución frágil de (P), encontramos condiciones suficientes para que $u^{-1}(D)$ tenga medida cero; esto es suficiente para que u sea una *solución fuerte* de (P): verifica (P) c.t.p. Mostramos condiciones para la existencia de extremos locales de I que son soluciones fuertes de (P). Finalmente, probamos que si f verifica una condición de crecimiento que involucra al primer valor propio de $-\Delta_p$, entonces (P) tiene una solución fuerte que globalmente minimiza I.
Palabras Clave funcional no-diferenciable, operador p-Laplaciano, problema de valor en la frontera, solución frágil, solución fuerte.

1 Introduction

In this paper we deal with the stationary counterpart of an equation having the form

$$\partial_t u(x,t) = -\Delta_p u(x,t) - G(x, u(x,t)), \tag{1}$$

where the nonlinear forcing term $G(x, \cdot)$ presents a countable number of downward or upward discontinuities. The model (1) helps to study the evolution of systems, (Vásquez, 2006), that present gradient-dependent diffusivity, i.e., a nonlinear diffusion phenomena described by the p-Laplace operator, $\Delta_p w = \mathrm{div}(|\nabla w|^{p-2} \nabla w)$. For $p = 2$, Δ_p coincides with the Laplace operator, $\Delta w = w_{x_1 x_1} + ... + w_{x_N x_N}$. An introduction to the properties of the p-Laplace can be found e.g. in (Lindqvist, 2019).

Then we are concerned with the equation $-\Delta_p u(x) = G(x, u(x))$, which serves to study problems of plasma physics (see e.g. (Ambrosetti & Turner, 1988), (Cimatti, 1979) and (Pavlenko & Potapov, 2018)), electrophysics (see e.g. (Potapov, 2014)), fluid mechanics (see e.g. (Ambrosetti & Struwe, 1989)), chemical kinetics (see e.g. (Frank-Kamenetskii, 1969)), astrophysics (see e.g. (Chandrasekar, 1985)), etc. In concrete we are interested in the case of $G(x,s) = \phi(x,s) + q(x)$ and $\phi(x,s) = h(x)f(s)$, i.e.,

$$\begin{cases} -\Delta_p u(x) = h(x)\, f(u(x)) + q(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{P}$$

where $p > 1$, $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with smooth boundary, $q \in \mathrm{L}^{p'}(\Omega)$, $1/p + 1/p' = 1$,

(H) $h \in \mathrm{L}^{\infty}(\Omega) \setminus \{0\}$,

and $f : \mathbb{R} \to \mathbb{R}$ verifies the following conditions.

(F1) There exists a countable set

$$D = D_b \cup D_d = \{b_1, ..., b_k, \dots\} \cup \{d_1, ..., d_k, \dots\} \subseteq \mathbb{R}$$

such that f is continuous on $\mathbb{R} \setminus D$ and, for each $k \in \mathbb{N}$,

$$f(b_k^-) < f(b_k^+), \quad f(b_k) \in \left[f(b_k^-),\ f(b_k^+)\right],$$

$$f(d_k^+) < f(d_k^-), \quad f(d_k) \in \left[f(d_k^+),\ f(d_k^-)\right].$$

(F2) There exist $\alpha, C_1, C_2 > 0$ such that $1 + \alpha \in [p, p^*]$ and $|f(s)| \leq C_1 + C_2\,|s|^\alpha$, for every $s \in \mathbb{R}$.

Remark 1. Here we denote $w(a^-) = \lim_{z\uparrow a} w(z)$ and $w(a^+) = \lim_{z\downarrow a} w(z)$. As usual, $p^* = Np/(N-p)$ if $p > N$, otherwise, $p^* = +\infty$. We shall also denote $\Omega_+ = \{x \in \Omega \,/\, h(x) \geq 0\}$, $\Omega_- = \{x \in \Omega \,/\, h(x) < 0\}$ and

$$F(s) = \int_0^s f(y)\, dy.$$

Remark 2. Along the document, the Sobolev space $\mathrm{W}_0^{1,p}(\Omega)$ shall be equipped with the norm given by $\|u\|^p_{\mathrm{W}_0^{1,p}(\Omega)} = \int_\Omega |\nabla u(x)|^p\, dx$ and, its dual space will be written $\mathrm{W}^{-1,p'}(\Omega)$.

Because of (F1), the standard functional associated to (P), $I : \mathrm{W}_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ given by

$$I(u) = \frac{1}{p}\int_\Omega |\nabla u(x)|^p dx - \int_\Omega q(x)u(x)dx - \int_\Omega h(x)\, F(u(x))\, dx,$$

is not Fréchet differentiable and, consequently, the usual variational methods can not be applied. However, as we will see, condition (F2) implies that I is locally Lipschitz and, therefore, for every $u \in \mathrm{W}_0^{1,p}(\Omega)$ there is a generalized gradient, (Clarke, 1990), given by $\partial I(u) = \left\{\xi \in \mathrm{W}^{-1,p'}(\Omega) \,/\ \ \forall v \in \mathrm{W}_0^{1,p}(\Omega) :\ I^0(u; v) \geq \langle \xi,\ v\rangle\right\}$, where the generalized directional derivatives are given by

$$I^0(u; v) = \limsup_{w\to u,\, \lambda\downarrow 0} \frac{I(w + \lambda v) - I(w)}{\lambda}.$$

In this context, $u \in \mathrm{W}_0^{1,p}(\Omega)$ is a *generalized critical point* of I iff $0 \in \partial I(u)$. For more details in the variational methods for non-differential functionals, the reader can check (Giaquinta, 1983).

Before stating our main results, let's introduce a multivalued function which shall be useful. Let $x \in \Omega$ and $s \in \mathbb{R}$. We put $\hat{\phi}(x, s) = \{h(x)f(s)\}$ if $s \notin D$; otherwise, there exists some $k \in \mathbb{N}$ such that $s = b_k$ or $s = d_k$, and one of the following points holds

$$\hat{\phi}(x, b_k) = \begin{cases} [h(x)f(b_k^-),\ h(x)f(b_k^+)], & \text{if } x \in \Omega_+, \\ [h(x)f(b_k^+),\ h(x)f(b_k^-)], & \text{if } x \in \Omega_-, \end{cases}$$

$$\hat{\phi}(x, d_k) = \begin{cases} [h(x)f(d_k^+),\ h(x)f(d_k^-)], & \text{if } x \in \Omega_+, \\ [h(x)f(d_k^-),\ h(x)f(d_k^+)], & \text{if } x \in \Omega_-. \end{cases}$$

Remark 3. Along the document we will have $s = u(x)$, where the function $u : \Omega \to \mathbb{R}$ is related to the problem (P). In this context, the following notation will be useful. For each $k \in \mathbb{N}$,

$$\Gamma_{b,k} = u^{-1}(\{b_k\}), \qquad \Gamma_{d,k} = u^{-1}(\{d_k\}),$$

$$\Gamma_b = u^{-1}(D_b) = \dot{\bigcup_{k=1}^{\infty}} \Gamma_{b,k}, \qquad \Gamma_d = u^{-1}(D_d) = \dot{\bigcup_{k=1}^{\infty}} \Gamma_{d,k}, \tag{2}$$

$$\Gamma = u^{-1}(D) = \Gamma_b \,\dot{\cup}\, \Gamma_d. \tag{3}$$

Therefore, we would have

$$\hat{\phi}(x, u(x)) = \begin{cases} \{h(x)f(u(x))\}, & \text{if } x \in \Omega \setminus \Gamma, \\ [h(x)f(u(x)^-),\ h(x)f(u(x)^+)], & \text{if } x \in \Omega_+ \cap \Gamma_b, \\ [h(x)f(u(x)^+),\ h(x)f(u(x)^-)], & \text{if } x \in \Omega_- \cap \Gamma_b, \\ [h(x)f(u(x)^+),\ h(x)f(u(x)^-)], & \text{if } x \in \Omega_+ \cap \Gamma_d, \\ [h(x)f(u(x)^-),\ h(x)f(u(x)^+)], & \text{if } x \in \Omega_- \cap \Gamma_d. \end{cases} \tag{4}$$

Our first main result, which shall be proved in Section 2, provides a characterization of *frail solutions* of (P) as generalized critical points of I:

Theorem 1. *Assume (F1), (F2) and (H). Then $u \in \mathrm{W}_0^{1,p}(\Omega)$ is a generalized critical point of I iff it's a frail solution of* (P)*, i.e., if it verifies*

$$-\Delta_p u(x) - q(x) \in \hat{\phi}(x, u(x)), \quad \textit{for a.e. } x \in \Omega. \tag{5}$$

In this case, it holds

$$-\Delta_p u(x) - q(x) = h(x)f(u(x)), \quad \textit{for a.e. } x \in \Omega \setminus \Gamma. \tag{6}$$

Our second main result, that will be proved in Section 3, provides a sufficient condition for $u \in \mathrm{W}_0^{1,p}(\Omega)$, a frail solution of (P), to be a *strong solution*, as used in (Potapov, 2014), that is, whenever

$$-\Delta_p u(x) = q(x) + h(x)f(u(x)), \quad \text{for a.e. } x \in \Omega.$$

For this we need the following notation:

$$m_+ = \operatorname*{ess\,inf}_{x\in\Omega_+}(h(x)), \quad M_+ = \operatorname*{ess\,sup}_{x\in\Omega_+}(h(x)), \tag{7}$$
$$m_- = \operatorname*{ess\,inf}_{x\in\Omega_-}(h(x)), \quad M_- = \operatorname*{ess\,sup}_{x\in\Omega_-}(h(x)),$$

$$Z_b = \bigcup_{k=1}^{\infty}\left[\alpha^-_{k,b}, \alpha^+_{k,b}\right], \quad Z_d = \bigcup_{k=1}^{\infty}\left[\alpha^-_{k,d}, \alpha^+_{k,d}\right], \tag{8}$$

where, for $k \in \mathbb{N}$,

$$\begin{aligned}
\alpha^-_{k,b} &= \min\left\{m_-\, f(b^+_k),\ M_-\, f(b^+_k),\ m_+\, f(b^-_k),\ M_+\, f(b^-_k)\right\},\\
\alpha^+_{k,b} &= \max\left\{m_-\, f(b^-_k),\ M_-\, f(b^-_k),\ m_+\, f(b^+_k),\ M_+\, f(b^+_k)\right\},\\
\alpha^-_{k,d} &= \min\left\{m_-\, f(d^-_k),\ M_-\, f(d^-_k),\ m_+\, f(d^+_k),\ M_+\, f(d^+_k)\right\},\\
\alpha^+_{k,d} &= \max\left\{m_-\, f(d^+_k),\ M_-\, f(d^+_k),\ m_+\, f(d^-_k),\ M_+\, f(d^-_k)\right\}.
\end{aligned} \tag{9}$$

Remark 4. Observe that $\|h\|_{L^\infty(\Omega)} = \max\{|m_-|,\ M_+\}$.

Theorem 2. *Assume (F1), (F2) and (H). Let $u \in W^{1,p}_0(\Omega)$ be a frail solution of* (P).

i) If $|\Gamma| = 0$, then u is a strong solution of (P).
ii) If $-q(x) \notin Z_b \cup Z_d$, for a.e. $x \in \Omega$, then $|\Gamma| = 0$.

Our last main result, which shall be proved in Section 4, provides sufficient conditions for a point of local extremum of I to be a strong solution of (P).

Theorem 3. *Assume (F1), (F2) and (H). Suppose that*

i) $u \in W^{1,p}_0(\Omega)$ is a point of local minimum of I, $|\Omega_-| = 0$ and $|\Gamma_d| = 0$ or,
ii) $u \in W^{1,p}_0(\Omega)$ is a point of local minimum of I, $|\Omega_+| = 0$ and $|\Gamma_b| = 0$ or,
iii) $u \in W^{1,p}_0(\Omega)$ is a point of local maximum of I, $|\Omega_+| = 0$ and $|\Gamma_d| = 0$ or,
iv) $u \in W^{1,p}_0(\Omega)$ is a point of local maximum of I, $|\Omega_-| = 0$ and $|\Gamma_b| = 0$.

Then $|\Gamma| = 0$ and, consequently, u is a strong solution of (P).

As a consequence, if f verifies a suitable growing condition involving the first eigenvalue of $-\Delta_p$, then (P) has a ground state, i.e., a strong solution which is a global minimizer of I. This is proved in Section 4.

As it was already mentioned, we prove Theorems 1, 2 and 3 in Sections 2, 3 and 4, respectively. Our main tools are the variational methods for non-differentiable functionals produced by Chang and Clarke, as presented e.g. in (Chang, 1981) and (Clarke, 1990). Our results extend those of (Calahorrano & Mayorga-Zambrano, 2001) where it's assumed that f has only one upward discontinuity and that, in addition to condition (H), h is bounded away from zero, $\operatorname*{ess\,inf}_{x\in\Omega} h(x) > 0$. The setting of (Arcoya & Calahorrano, 1994) is easier than that of (Calahorrano & Mayorga-Zambrano, 2001) as the authors consider $h \equiv 1$ in Ω.

2 Characterization of frail solutions

In this section we prove that frail solutions of (P) are generalized critical points of I, provided conditions (F1), (F2), and (H) hold.

To start with, let's observe that, by (F2), (H) and Hölder-Minkowski inequality, (Chang, 1981), the functional $\tilde{J} : \mathrm{L}^{\alpha+1}(\Omega) \to \mathbb{R}$, given by

$$\tilde{J}(u) = \int_\Omega \int_0^{u(x)} \phi(x,s)\, ds dx = \int_\Omega h(x)\, F(u(x))\, dx,$$

verifies

$$|\tilde{J}(u) - \tilde{J}(v)| \leq \|h\|_{\mathrm{L}^\infty(\Omega)} \left[C_1\, |\Omega|^{\alpha/(\alpha+1)} + C_2 \sup_{w \in U} \|w\|^{\alpha/(\alpha+1)}_{\mathrm{L}^{\alpha+1}(\Omega)} \right] \|u - v\|_{\mathrm{L}^{\alpha+1}(\Omega)}\,,$$

for all $u, v \in U$, where U is any open bounded subset of $\mathrm{L}^{\alpha+1}(\Omega)$; so that $\tilde{J}$ is locally Lipschitz. Since the immersion $\mathrm{W}^{1,p}_0(\Omega) \subseteq \mathrm{L}^{\alpha+1}(\Omega)$ is dense and continuous, it follows, (Chang, 1981, Th.2.2&2.3) and (Chang, 1981, Cor. pp 111), that

$$\partial J(u) \subseteq \partial \tilde{J}(u) \subseteq \hat{\phi}(\cdot, u(\cdot)), \qquad \text{a.e. in } \Omega, \tag{10}$$

where J denotes the restriction of $\tilde{J}$ to $\mathrm{W}^{1,p}_0(\Omega)$ and it is used the identification $\mathrm{L}^{(\alpha+1)/\alpha}(\Omega) \cong (\mathrm{L}^{\alpha+1}(\Omega))^*$.

Remark 5. It's important to note (see (Clarke, 1990, pg. 54, 55) and (Chang, 1981, Prop. 3 & 4, pg. 104)) that if $\beta \in \mathbb{R}$ and $B, H : E \to \mathbb{R}$ are locally Lipschitz functionals on a Banach space E, then for every $y \in E$, $\partial B(\beta y) = \beta\, \partial B(y)$ and $\partial(B + H)(y) \subseteq \partial B(y) + \partial H(y)$. Moreover, if H has a continuous Gateaux derivative H'_G, then $\partial H(y) = \left\{H'_G(y)\right\}$, for every $y \in E$. Recall also that Fréchet differentiablility implies Gateaux differentiability.

Proof (of Theorem 1). For $u \in \mathrm{W}^{1,p}_0(\Omega)$, we have that $I(u) = Q(u) - J(u) + R(u)$, where

$$Q(u) = \frac{1}{p} \int_\Omega |\nabla u(x)|^p\, dx, \qquad R(u) = -\int_\Omega q(x) u(x) dx.$$

Since Q and R are Fréchet differentiable, we get, by Remark 5 and point (10), that

$$\partial I(u) = \{Q'(u)\} - \partial J(u) + \{R'(u)\}, \qquad \partial J(u) \subseteq \partial \tilde{J}(u) \subseteq \hat{\phi}(\cdot, u(\cdot)), \qquad \text{a.e. in } \Omega.$$

By definition, $u \in \mathrm{W}^{1,p}_0(\Omega)$ is a generalized critical point of I if and only if $0 \in \partial I(u)$ which, in its turn, it is equivalent to the existence of $\omega \in \partial J(u)$ such that,

$$Q'(u) - \omega + R'(u) = 0 \quad \text{and} \qquad \omega(x) \in \hat{\phi}(x, u(x)), \qquad \text{for a.e. } x \in \Omega, \tag{11}$$

where we are considering ω both as a function in $\mathrm{L}^{(\alpha+1)/\alpha}(\Omega) \cong (\mathrm{L}^{\alpha+1}(\Omega))^*$ and as a functional living in $(\mathrm{L}^{\alpha+1}(\Omega))^* \subseteq \mathrm{W}^{-1,p'}(\Omega)$. Therefore, for all $v \in \mathrm{W}^{1,p}_0(\Omega)$ it holds $\langle Q'(u) + R'(u), v\rangle = \langle \omega, v\rangle$, i.e.,

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla v(x)dx - \int_\Omega q(x)v(x)dx = \int_\Omega w(x)v(x)dx. \tag{12}$$

By the arbitrariness of v and the isomorphisms recently mentioned, we get $-\Delta_p u = w + q \in \mathrm{L}^{(\alpha+1)/\alpha}(\Omega)$ and $-\Delta_p u(x) = w(x) + q(x)$, for a.e. $x \in \Omega$, so that, by (11),

$$-\Delta_p u(x) - q(x) \in \hat{\phi}(x, u(x)), \qquad \text{for a.e. } x \in \Omega,$$

which, thanks to (4), implies (5). ■

3 Existence of strong solutions

In this section, we prove Theorem 2, i.e. that a frail solution becomes strong if the image of $-q$ does not intersect, a.e., the intervals $[\alpha^-_{k,b}, \alpha^+_{k,b}]$ and $[\alpha^-_{k,d}, \alpha^+_{k,d}]$.

Before this, it's worth mentioning that the type of results like Theorem 2 appeared first in (Ambrosetti & Badiale, 1989). There it is considered the case of $p = 2$, $h \equiv 1$, f having only one upward discontinuity, and it is required the existence of some $m > 0$ such that the function with formula $f(s) + m\,s$ is increasing. Instead of dealing with the non-differentiable functional I, the authors applied the classical critical point theory to a dual functional Ψ, of class C^1 on $\mathrm{L}^2(\Omega)$. It seems unlikely that their technique, *Clarke's dual principle*, could be brought to the context of Theorem 2 because the first term in (12) is not linear in u.

Proof (of Theorem 2). Let's recall that $u \in W^{1,p}_0(\Omega)$ is a frail solution of (P). Let's assume that

$$-q(x) \notin Z_b \cup Z_d, \qquad \text{for a.e. } x \in \Omega, \tag{13}$$

where Z_b and Z_d are given in (8). Then, by Theorem 1, it verifies,

$$-\Delta_p u(x) - q(x) \in \hat{\phi}(x, u(x)), \qquad \text{for a.e. } x \in \Omega. \tag{14}$$

For each $k \in \mathbb{N}$ we have that $u(x) = b_k$, if $x \in \Gamma_{b,k}$, and $u(x) = d_k$, if $x \in \Gamma_{d,k}$. Therefore, by (Morrey, 2008, Th. 3.2.2), it follows that

$$\Delta_p u(x) = 0, \quad x \in \Gamma_{b,k} \cup \Gamma_{d,k}.$$

By (2) and (3), we get $\Delta_p u(x) = 0$, $x \in \Gamma$, which, together with (14), imply that $-q(x) \in \hat{\phi}(x, u(x))$, for a.e. $x \in \Gamma$, i.e., by considering (4),

$$-q(x) \in \begin{cases} [h(x)f(u(x)^-),\ h(x)f(u(x)^+)], & \text{for a.e. } x \in \Omega_+ \cap \Gamma_b, \\ [h(x)f(u(x)^+),\ h(x)f(u(x)^-)], & \text{for a.e. } x \in \Omega_- \cap \Gamma_b, \\ [h(x)f(u(x)^+),\ h(x)f(u(x)^-)], & \text{for a.e. } x \in \Omega_+ \cap \Gamma_d, \\ [h(x)f(u(x)^-),\ h(x)f(u(x)^+)], & \text{for a.e. } x \in \Omega_- \cap \Gamma_d, \end{cases}$$

whence $-q(x) \in Z_b \cup Z_d$, for a.e. $x \in \Gamma$. Therefore, point (13) implies that $|\Gamma| = 0$. The last, together with (6), produce

$$-\Delta_p u(x) - q(x) = h(x) f(u(x)), \quad \text{for a.e. } x \in \Omega,$$

i.e., u is a strong solution of (P). ■

4 Existence of extremum strong solutions

As it was mentioned before, in this section we prove Theorem 3, which provides sufficient conditions for a point of local maximum or minimum of the functional I to be a strong solution of (P).

Proof (of Theorem 3). By (Clarke, 1990, Prop. 2.3.2), any point of local extremum of I is a generalized critical point of I so that, by Theorem 1, it is a frail solution of (P). Then, by following part of the scheme for proving Theorem 2, we get

$$-q(x) \in Z_b \cup Z_d, \quad \text{for a.e. } x \in \Gamma. \tag{15}$$

Let us recall that, by (4), we have $\hat{\phi}(x, u(x)) = \{h(x) f(u(x))\}$, $x \in \Omega \setminus \Gamma$, so that point (6) holds:

$$-\Delta_p u(x) - q(x) = h(x) f(u(x)), \quad \text{for a.e. } x \in \Omega \setminus \Gamma. \tag{16}$$

We will prove only point i) as the cases ii), iii), and iv) are handled in a similar way. Then let us assume that

$$|\Omega_-| = |\Gamma_d| = 0 \tag{17}$$

and that $u \in W_0^{1,p}(\Omega)$ is a point of local minimum of I. Thanks to Theorem 2, to obtain the result it's enough to show that $|\Gamma| = 0$. From (17) it follows that

$$|\Gamma| = \sum_{k=1}^{\infty} |\Gamma_{b,k}^+|, \qquad \Gamma_{b,k}^+ = \Gamma_{b,k} \cap \Omega_+. \tag{18}$$

On the other hand, for each $k \in \mathbb{N}$ we have, by (15) and (8), that

$$|\Gamma_{b,k}^+| \leq |\{x \in \Gamma_{b,k}^+ \ / \ -q(x) \neq \alpha_{k,b}^-\}| + |\{x \in \Gamma_{b,k}^+ \ / \ -q(x) \neq \alpha_{k,b}^+\}|. \tag{19}$$

Let us prove that

$$\forall k \in \mathbb{N}: \quad |\{x \in \Gamma_{b,k}^+ \ / \ -q(x) \neq \alpha_{k,b}^+\}| = 0. \tag{20}$$

Let us reason by *reductio ad absurdum*. Then let us assume that for some $k_0 \in \mathbb{N}$,

$$|\{x \in \Gamma_{b,k_0}^+ \ / \ -q(x) \neq \alpha_{k_0,b}^+\}| > 0.$$

Let's pick a positive function $\psi \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$. Since u is a point of local minimum for I, there exists $\tilde{\varepsilon} > 0$ such that $I(u) \leq I(u + \varepsilon\psi)$, for every $\varepsilon \in]0, \tilde{\varepsilon}[$. By direct computation, having in consideration (16), (17), (18) and (9), we get

$$
\begin{aligned}
0 &\leq \lim_{\varepsilon \downarrow 0} \frac{I(u+\varepsilon\psi) - I(u)}{\varepsilon} = \langle Q'(u), \psi \rangle + \langle R'(u), \psi \rangle - \lim_{\varepsilon \downarrow 0} \frac{J(u+\varepsilon\psi) - J(u)}{\varepsilon} \\
&= \int_{\Omega_+ \cap \Gamma_b} |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla \psi(x)\, dx - \int_{\Omega_+ \cap \Gamma_b} q(x)\psi(x)\, dx \\
&\quad - \int_{\Omega_+ \cap \Gamma_b} h(x)\, f(u(x)^+)\, \psi(x)\, dx, \\
&= \int_{\Omega_+ \cap \Gamma_b} \left[-\Delta_p u(x) - q(x) - h(x) f(u(x)^+) \right] \psi(x)\, dx \\
&= \sum_{k=1}^{\infty} \int_{\Gamma_{b,k}^+} \left[-\Delta_p u(x) - q(x) - h(x) f(b_k^+) \right] \psi(x)\, dx \\
&\leq \int_{\Gamma_{b,k_0}^+} \left[-q(x) - h(x) f(b_{k_0}^+) \right] \psi(x)\, dx \\
&< \int_{\{x \in \Gamma_{b,k_0}^+ \,/\, -q(x) < \alpha_{k_0,b}^+\}} \left[\alpha_{b,k_0}^+ - h(x) f(b_{k_0}^+) \right] \psi(x)\, dx \leq 0,
\end{aligned}
$$

which is a contradiction; so that (20) is true.

In a similar way it is proved that

$$
\forall k \in \mathbb{N} : \quad |\{x \in \Gamma_{b,k}^+ \,/\, -q(x) \neq \alpha_{k,b}^-\}| = 0. \tag{21}
$$

Therefore, by (18), (19), (20) and (21), it follows that $|\Gamma| = 0$. ∎

As a consequence of Theorem 3 we next show that if the following condition, which involves the first eigenvalue of $-\Delta_p$, holds, then (P) has a *ground state*, i.e., a strong solution which is a global minimizer of I.

(F3) There exist $\delta, \rho > 0$, with $\delta < \lambda_1 / \|h\|_{L^\infty(\Omega)}$, such that

$$
\forall s \in \mathbb{R} : \quad |f(s)| \leq \delta |s|^{p-1} + \rho,
$$

where λ_1 is the first eigenvalue of $-\Delta_p$.

Corollary 1. *Assume (F1), (F3) and (H). If* $|\Omega_+| = |\Gamma_b| = 0$ *or* $|\Omega_-| = |\Gamma_d| = 0$*, then problem* (P) *has a ground state.*

Proof. Since $0 < \lambda_1 = \inf \left\{ \int_\Omega |\nabla u(x)|^p dx \,/\, u \in W_0^{1,p}(\Omega) \wedge \|u\|_{L^p(\Omega)} = 1 \right\}$, (Cimatti, 1979), we have that

$$
\forall u \in W_0^{1,p}(\Omega) \setminus \{0\} : \quad \lambda_1 \int_\Omega |u(x)|^p dx \leq \int_\Omega |\nabla u(x)|^p dx. \tag{22}
$$

Let $x \in \Omega$ and $u \in \mathrm{W}_0^{1,p}(\Omega)$. By (F3) we have that

$$\begin{aligned} |h(x)F(u(x))| &\le \|h\|_{\mathrm{L}^\infty(\Omega)} |F(u(x))| \le \|h\|_{\mathrm{L}^\infty(\Omega)} \int_0^{u(x)} |f(s)| ds \\ &\le \|h\|_{\mathrm{L}^\infty(\Omega)} \left(\delta \int_0^{u(x)} |s|^{p-1} ds + \rho \int_0^{u(x)} ds \right) \\ &\le \frac{\delta \cdot \|h\|_{\mathrm{L}^\infty(\Omega)}}{p} |u(x)|^p + \|h\|_{\mathrm{L}^\infty(\Omega)} \rho |u(x)|, \end{aligned} \tag{23}$$

whence, by using (22) and Hölder inequality,

$$\begin{aligned} \int_\Omega |h(x)F(u(x))| dx &\le \frac{\delta \cdot \|h\|_{\mathrm{L}^\infty(\Omega)}}{p\lambda_1} \int_\Omega |\nabla u(x)|^p dx + \|h\|_{\mathrm{L}^\infty(\Omega)} \rho \int_\Omega |u(x)| dx \\ &\le \frac{\delta \cdot \|h\|_{\mathrm{L}^\infty(\Omega)}}{p\lambda_1} \|\nabla u\|_{\mathrm{L}^p(\Omega)}^p + \|h\|_{\mathrm{L}^\infty(\Omega)} \rho |\Omega|^{1/p'} \|u\|_{\mathrm{L}^p(\Omega)} . \end{aligned} \tag{24}$$

Therefore, by (24), we get

$$\begin{aligned} I(u) &= \frac{1}{p} \int_\Omega |\nabla u(x)|^p \, dx - \int_\Omega h(x)F(u(x)) dx - \int_\Omega q(x)u(x) dx \\ &\ge \frac{1}{p} \|\nabla u\|_{\mathrm{L}^p(\Omega)}^p - \int_\Omega |h(x)F(u(x))| dx - \|q\|_{\mathrm{L}^{p'}(\Omega)} \|u\|_{\mathrm{L}^p(\Omega)} . \\ &\ge \frac{1 - \delta \cdot \|h\|_{\mathrm{L}^\infty(\Omega)} / \lambda_1}{p} \|\nabla u\|_{\mathrm{L}^p(\Omega)}^p - k \|u\|_{\mathrm{L}^p(\Omega)} , \end{aligned} \tag{25}$$

where $k = \|q\|_{\mathrm{L}^{p'}(\Omega)} + \|h\|_{\mathrm{L}^\infty(\Omega)} \rho |\Omega|^{1/p'} > 0$. Since $1 - \delta \cdot \|h\|_{\mathrm{L}^\infty(\Omega)} / \lambda_1 > 0$ and $p > 1$, it follows from (25) and Poincaré's inequality, (Brezis, 2010, Cor.9.10), that $I(u) \longrightarrow +\infty$, as $\|u\|_{\mathrm{W}_0^{p,1}(\Omega)} \longrightarrow +\infty$, i.e., I is coercive.

By using the dominated convergence theorem, (Brezis, 2010, Th.4.2), $-J$ is weakly lower semicontinuous; actually

$$\int_\Omega \int_0^{u_m(x)} h(x) f(s) ds dx \longrightarrow \int_\Omega \int_0^{u(x)} h(x) f(s) ds dx, \tag{26}$$

as $u_m \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$. In fact, we may assume (perhaps extracting a subsequence) that $u_m \longrightarrow u$ a.e. in Ω and, with this, u_m is bounded, i.e., there exists $\Theta > 0$ such that $|u_m(x)| \le \Theta$, for a.e. $x \in \Omega$ and every $m \in \mathbb{N}$. Then, by (23), we have, for a.e. $x \in \Omega$,

$$|\Phi_m(x)| \le \frac{\delta \cdot \|h\|_{\mathrm{L}^\infty(\Omega)}}{p} \Theta^p + \|h\|_{\mathrm{L}^\infty(\Omega)} \rho \Theta = g(x),$$

where $\Phi_m(x) = h(x)F(u_m(x))$, and $g \in \mathrm{L}^1(\Omega)$.

Consider $\Phi(x) = h(x)F(u(x))$. Then

$$|\Phi_m(x) - \Phi(x)| \leq \|h\|_{L^\infty(\Omega)} \left| \int_{u(x)}^{u_m(x)} f(s) ds \right| \leq \|h\|_{L^\infty(\Omega)} \left| \int_{u(x)}^{u_m(x)} \left(\delta |s|^{p-1} + \rho\right) ds \right|$$
$$\leq \|h\|_{L^\infty(\Omega)} \left[\frac{\delta}{p} |u_m(x)|^p + \rho |u_m(x)| - \frac{\delta}{p} |u(x)|^p - \rho |u(x)| \right].$$

Since $u_m \longrightarrow u$ a.e. in Ω, we get that $\Phi_m(x) \longrightarrow \Phi(x)$, as $m \to +\infty$, for a.e. $x \in \Omega$. Then, by the dominated convergence theorem, we get (26).

The weakly lower semicontinuity of $-J$, together with the differentiablity of Q and R, imply that I is weakly lower semicontinuous. The last together with the coerciveness of I imply, (Struwe, 2008, Th.1.2), that there exists $u \in W_0^{1,p}(\Omega)$, a point of global minimum. Finally, by applying i) or ii) of Theorem 3, we conclude. ∎

Acknowledgments. The authors thank Yachay Tech community for its support during the whole project.

Funding. This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Conflicts of interest/Competing interests. The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Availability of data and material. Not applicable.

Code availability. Not applicable.

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