Matemática: Una publicación de FCNM – ESPOL
2018, Vol.16, No.2 **Publicado:2018-10-02**

SOLUTION FOR INHOMOGENEOUS SECOND ORDER ELLIPTIC EQUATIONS IN CLIFFORD-TYPE ALGEBRAS

SOLUCIÓN PARA ECUACIONES ELÍPTICAS DE SEGUNDO ORDEN NO HOMOGÉNEAS EN ALGEBRAS DE TIPO CLIFFORD

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Resumen*:* *En el presente trabajo, nosotros usamos álgebras de Clifford generalizadas llamadas álgebras de tipo Clifford o álgebras dependiendo de parámetros, para obtener soluciones distribucionales de la ecuación $\tilde{\Delta}=h$, donde h es una función integrable y $\tilde{\Delta}$ es conocido como el Laplaciano en el álgebra paramétrica o simplemente Laplaciano generalizado. Para hacer esto, nosotros usamos el Kernel de Newton paramétrico $\tilde{K}(x,\xi)$ y la formula integral de Cauchy-Pompeiu para el Laplaciano generalizado. También, nosotros discutimos un método para combinar operadores de primer y segundo orden, con la finalidad de obtener fórmulas de representaciones integrales para operadores con potencias.*
Adicionalmente, nosotros discutimos brevemente otras posibles aplicaciones en física, donde la solución resultante del asociado a la ecuación de Laplace modificada se puede interpretar en física de la materia condensada, en propiedades de transporte de los problemas de Dirac Fermion. Nosotros conjeturamos que, nuestras soluciones podrían ser relevantes en el análisis de nuevas fases exóticas de la naturaleza. Como, aislantes tipológicos donde, la naturaleza de Dirac de los portadores de carga implica nuevas propiedades físicas que van más allá de la descripción estándar de los portadores de carga convencionales en sistemas electrónicos por medio de la ecuación de Schrödinger.
Palabras Claves: *Soluciones distribucionales, Operadores elípticos, Álgebras de tipo Clifford, Formulas de Representación integral.*

Abstract: *In the present work we use a generalized Clifford algebras called Clifford Type algebras o Clifford parametric algebras in order to obtain a distributional solution for the inhomogeneous equations $\tilde{\Delta}=h$, where h is an integrable function, $\tilde{\Delta}$ is the Laplacian in the Clifford parametric algebras or simply generalized Laplacian. To do that, we use the parameter Newton kernel $\tilde{K}(x,\xi)$ and the Cauchy-Pompeiu integral formula for the generalized Laplacian. We also discuss a method to combine operators of first and second order in order to obtain integral representation formulas for higher order operators.*
In addition, we briefly mention some other possible physical realizations, were the solutions to the resulting modified Laplace equation can be interpreted in condensed matter physics specifically, in the transport properties of Dirac fermion problems. We conjecture that our solutions could be relevant in the analysis of new exotic phases of nature, such as topological insulators were the Dirac nature of the charge carriers implies new physical properties which go beyond the standard description of conventional charge carriers in electronic systems by means of the Schrödinger equation.
Keywords*:* *Distributional solutions, elliptic operators, Clifford-type algebras, integral representations formulas.*

1 INTRODUCTION

Currently, the development of new mathematical tools to describe newly discovered phases of nature has been given a great deal of attention in the physics literature. For instance, the authors of reference [1] used the Clifford algebra to characterize topological phases of matter. This work has been followed by several other research papers considering, for example, the Classification of stable three-dimensional Dirac semimetals with nontrivial topology by means of the Clifford algebraic properties [2]. One interesting point to be remarked is that prior to these developments, the studies of Clifford algebras in the physical real had mostly been focused on the definition of the Dirac equation and no further properties of the Clifford algebras were exploited.

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In a previous work, we have argued on the feasibility of considering modified Clifford algebras to describe physical phenomena [12]. In this present paper we discuss briefly a physical example, within the scattering context, that would allow motivating the use of these new algebraic structures that might have applications in the realm of newly discovered exotic phases of nature such as topological insulators, anyons and modified algebraic Lie structures.

1.1 CLIFFORD-TYPE ALGEBRAS

Complex analysis is one of the most influential areas in mathematics. It has consequences in many branches of science such as algebra, geometry, number theory, potential theory, differential equations, dynamical systems, integral equations, integral transformations, harmonic analysis, global analysis, operator theory, algebraic geometry and many others. It also has many applications e.g. in physics. Classical ones are elasticity theory, fluid dynamics, shell theory, underwater acoustics and quantum mechanics. Complex analysis is in fact a simple accessible theory with more relation to other subjects in mathematics than other topics. In complex analysis all structural concepts in mathematics are stressed. Algebraic,

analytic and topological concepts occur and even geometry is involved. It also addresses questions of ordering sets that may be discussed in connection with complex analysis. Gauss, Cauchy, Weierstraßand Riemann were the main initiators of complex analysis and there was more than a century of rapid development. See [6, 8, 9].
In order to extend the complex idea to higher dimensions, one has two possibilities:
I) One considers a real Euclidean space $\mathbb{R}^{2n}$ of dimension $2n$, and one connects the $2n$ real variables $x_1, y_1, \dots, x_n, y_n$ to n complex variables $z_1, \dots, z_n$, where we have $z_j = x_j + iy_j$, $j = 1, \dots, n$. See [7, 16].
II) It is also possible to consider the real Euclidean space $\mathbb{R}^{1+n}$ of dimension $1+n$. In this case one can connect the $1+n$ real variables $x_0, x_1, \dots, x_n$ to a hyper-complex variable $x = x_0 + x_1e_1 + \cdots + x_ne_n$, where the basis vectors $e_0, e_1, \dots, e_n$ satisfy the relations

$$e_0 = 1, e_j^2 = -1, \text{ and } e_je_k + e_ke_j = 0$$

for $j, k = 1, \dots, n$ and $j \neq k$.

Clifford algebras can be defined as equivalence classes in the ring $\mathscr{R}[X_1, \dots, X_n]$ of polynomials in n variables $X_1, \dots, X_n$ with real coefficients, where two polynomials are to be distinguished in case they differ at least in the order of the factors X_j in one of their terms. Then two polynomials are said to be equivalent if their difference is a polynomial for which each term contains at least one of the factors

$$X_j^2 + 1 \quad \text{or} \quad X_iX_j + X_jX_i. \qquad (1.1)$$

Denoting X_j by e_j, $j = 1, \dots, n$, the structure polynomials (1.1) imply the well-known rules of the usual Clifford algebra $\mathscr{A}_n$;

$$e_j^2 = -1, \; j = 1, \dots, n, \quad \text{and}$$
$$e_ie_j = -e_je_i \quad \text{for } i \neq j.$$

It is well known that $\mathscr{A}_n$ extends the Euclidean space $\mathbb{R}^{1+n}$, which has the basis $e_0 = 1, e_1, \dots, e_n$ (see [10, 20]). Similarly, we can obtain Clifford type algebras if the structure polynomials (1.1) are replaced by

$$X_j^{kj} + \alpha_j \quad \text{and} \quad X_iX_j + X_jX_i - 2\gamma_{ij},$$

where $i, j = 1, \dots, n$, $i \neq j$, the $k_j \geq 2$ are natural numbers and α_j and $\gamma_{ij} = \gamma_{ji}$, for $i \neq j$ are real constants (see [20]). We call them Clifford algebras depending on parameters α_j and γij. If n≥2, the Clifford algebra generated by the structure polynomials (1.2) is denoted by $\mathscr{A}_n(k_j, \alpha_j, \gamma_{ij})$. For $n = 1$ we write $\mathscr{A}_1(k, \alpha)$. If in $\mathscr{A}_n(k_j, \alpha_j, \gamma_{ij})$ all of the k_j are equal to 2, we denote the obtained Clifford algebra by $\mathscr{A}_n(2, \alpha_j, \gamma_{ij})$. Like the usual Clifford algebra $\mathscr{A}_n(2, 1, 0)$, this Clifford algebra has the dimension 2^n.

Example. For instance, $\mathscr{A}_2(3, 1, 0)$ has dimension 9 and its basis is

$$1, e_1, e_2, e_1^2, e_1e_2, e_2^2, e_1^2e_2, e_1e_2^2, e_1^2e_2^2.$$

Moreover, $\mathscr{A}_2(k_j, 1, 0)$, with $k_1 = 2$ and $k_2 = 3$, has dimension 6, whereas its basis reads

$$1, e_1, e_2, e_1e_2, e_2^2, e_1e_2^2.$$

The Clifford algebra is $\mathscr{A}_n$(2,1,0).

In summary, a generalization of the Clifford algebra or simply **Clifford-type algebra** is given by fixing a set of real valued functions $\alpha_j(p)$ and $\gamma_{ij}(p), i, j = 1, \dots, n, i \neq j$, possibly depending on a parameter p, and considering the more general multiplication rules for the elements of the basis $\{e_A\}$. See $[20, 21]$

1.2 MONOGENIC FUNCTIONS IN $\mathscr{A}_n(2, \alpha_j, \gamma_{ij})$

The monogenic concept extends the ideas of holomorphic functions for standard complex algebras. It is important to say that there exist different ways of defining monogenecity. However, in these paper we use the usual definition provided by F.
Brackx, R. Delanghe and F. Sommen [10], despite that we mention a more general way to define monogenic as an example. Other books that you can consult are [13, 19].

1.2.1 CLIFFORD NUMBER

Let

$$x = x_0 + \sum_{j=1}^{n} x_je_j$$

be a point of $\mathbb{R}^{n+1}$. See $[10, 14, 15]$. To this point $\bar{x}$ by

$$\bar{x} = x_0 - \sum_{j=1}^{n} x_je_j.$$

x is called a "Clifford number" and x the "conjugate" Clifford number.

The definition of $\bar{x}$ implies

$$x\bar{x} = |x|^2.$$

1.2.2 CLIFFORD VALUED FUNCTION

Let Ω be an open and connected domain in $\mathbb{R}^{n+1}$ whose points will be denoted by $x = (x_0, x_1, \ldots, x_n)$. Let, further, $u(x)$ be a function with values in $\mathcal{A}_n$ defined in Ω. Denoting the real-valued components of $u(x)$ by $u_A(x)$, that is,

$$u(x) = \sum_A u_A(x) e_A,$$

where A is a set of index combination. See [10].
Example. A Clifford valued function in $\mathcal{A}_3$ is given by:

$$\sum_{i=0}^{3} u_i(x)e_i = u_{12}(x)e_{12} + u_{13}(x)e_{13} + u_{23}(x)e_{23} + u_{123}(x)e_{123},$$

where the basis is

$$\{e_0 = 1, e_1, e_2, e_3, e_{12}, e_{23}, e_{13}, e_{123}\}$$

Note that $\mathcal{A}_3$ is constructed over $\mathbb{R}^4$. Thus, each $uA(x) = uA(x0, x1, x2, x3)$, where A = {0, 1, 2, 3, 12, 13, 23, 123}.
Now, We will consider functions u defined on a bounded domain Ω in $\mathbb{R}^{n+1}$ having values in $\mathcal{A}_n$ $(2, \alpha_i, \gamma_{ij})$, i.e., $u(x) = \sum_{A\in\Gamma n} u_A(x)eA$, where $u_A: \Omega \to R$, and $A \in \Gamma_n = \{0, 1, 2, \ldots, 12, 13, \ldots, 123\ldots n\}$. The generalized Cauchy-Riemann operator and its conjugate in $\mathbb{R}^{n+1}$ are defined to be

$$D = \sum_{j=0}^{n} e_j \partial_j \ \ and \ \ \bar{D} = \partial_0 - \sum_{j=1}^{n} e_j \partial_j$$

Definition1.2. An $\mathcal{A}_n(2, \alpha_i, \gamma_{ij})$ −valued function u is called left monogenic if it satisfies the equation $Du = 0$ and right monogenic if it satisfies $uD = 0$. A left monogenic function will be called monogenic.

Example. Let $u = u_0 + u_1 e_1 + u_2 e_2 + u_{12} e_{12}$ be a monogenic function with values in $\mathcal{A}_2$. Then, the real-valued components u_0, u_1, u_2 and u_{12} satisfy the following system of differential equations:

$$\begin{aligned} \partial_0 u_0 - \partial_1 u_1 - \partial_2 u_2 &= 0, \\ \partial_0 u_1 + \partial_1 u_0 + \partial_2 u_{12} &= 0, \\ \partial_0 u_2 - \partial_1 u_{12} + \partial_2 u_0 &= 0, \\ \partial_0 u_{12} + \partial_1 u_2 - \partial_2 u_1 &= 0, \end{aligned}$$

Indeed, the basic elements of $\mathcal{A}_2$ is $\{e_0 = 1, e_1, e_2, e_{12}\}$ over $\mathbb{R}^3$. Carrying out the multiplication

$$D_u = (\partial_0 + e_1\partial_1 + e_2\partial_2)(u_0 + u_1 e_1 + u_2 e_2 + u_{12} e_{12})$$

and taking into account the relations

$$\begin{aligned} &e_1 e_1 = e_2 e_2 = -1 \\ &e_1 e_{12} = e_1(e_1 e_2) = -e_2 \\ &e_2 e_1 = -e_1 e_2 \\ &e_2 e_{12} = e_2(-e_2 e_1) = e_1, \end{aligned}$$

the equation $Du = 0$ leads to the above equations because the basic elements $1, e_1, e_2, e_{12}$ are linearly independent.

Example. In $\mathcal{A}_2$ $(x|2; \alpha_1; \alpha_2; \alpha_3)$ the Dirac system $Du = 0$, for a Clifford-algebra-valued function $u = u_0 + u_1 e_1 + u_2 e_2 + u_{12} e_{12}$, reads

$$\begin{aligned} \partial_0 u_0 - \alpha_1 \partial_1 u_1 + 2\gamma_{12}\partial_2 u_1 - \alpha_2 \partial_2 u_2 &= 0, \\ \partial_0 u_1 + \partial_1 u_0 + \alpha_2 \partial_2 u_{12} &= 0, \\ \partial_0 u_2 - \alpha_1 \partial_1 u_{12} + \partial_2 u_0 + 2\gamma_{12}\partial_2 u_{12} &= 0, \\ \partial_0 u_{12} + \partial_1 u_2 - \partial_2 u_1 &= 0, \end{aligned}$$

Finally using D and D, we can define the Laplace operator as

$$\bar{D}D = D\bar{D} = \Delta_{n+1} = \Delta.$$

An important consequence of the monogenic function is the fact that every monogenic function is solution of the Laplace equation. Suppose that the function $u(x) = \sum_A u_A(x)eA$ is twice continuously differentiable and monogenic. Using the operator Δ, we obtain for u:

$$\Delta_{n+1} u = \Delta_{n+1} \sum_A u_A(x) e_A = \sum_A \Delta_{n+1} u_A(x) e_A.$$

In view of the linear independence of the basis elements e_A the equation $\Delta_{n+1} u = 0$ implies $\Delta_{n+1} u_A(x) = 0$ for each u_A. Hence, the following statement has been proved:

Lemma 1.3. The real-valued components u_A of a monogenic function are solutions of the Laplace equation $\Delta_{n+1} u_A(x) = 0$.

Since the Clifford algebra $\mathcal{A}_n$ has 2^n basis elements, a $\mathcal{A}_n$-valued function has 2^n real components.
In addition, the equation Du has 2^n real components. Therefore, the equation $Du = 0$ can be decomposed into 2^n real equations. These equations form a system of 2^n linear

partial differential equations of first order for the 2^n real components of u.

2 SYMMETRIC GREEN INTEGRAL FORMULA FOR THE OPERATOR $\tilde{\Delta}$

Let Ω be a bounded domain in $\mathbb{R}^{n+1}$ with sufficiently smooth boundary. Consider an $\mathcal{A}_n(2,\alpha_i,\gamma_{ij})$ −valued function $u=\sum_A u_A e_A$ defined in Ω and suppose it is monogenic, then we get the homogeneous second order differential equation

$$\tilde{\Delta}u=\bar{D}Du=\partial_0^2 u+\sum_{j=1}^{n}\alpha_j\partial_j^2 u-2\sum_{i<j}\gamma_{ij}\partial_i\partial_j u=0 \tag{2.2}$$

Since the coefficients α_j and $\gamma_{ij}=\gamma_{ji}$ are real, each real-valued component u_A of u satisfies this differential equation. If the α_j are supposed to be positive and the absolute values of the γ_{ij} are not too large then (2.1) is elliptic.

In order to write a symmetric Green Integral Formula for the operator $\tilde{\Delta}$, we rewrite the operator $\tilde{\Delta}$ as:

$$\tilde{\Delta}=\sum_{i,j}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_i}\right),$$

where, $a_{00}(x)=1, a_{ii}(x)=\alpha_i>0;\ \forall_i=1,\cdots n,\ a_{0j}(x)=a_{i0}(x)=0,\ a_{ij}(x)=-\gamma_{ij}, a_{ji}(x)=-\gamma_{ji}$ for $i\neq j,\ i,j=1,\cdots,n$ and $\gamma_{ij}=\gamma_{ji}$. Since the coefficients a_{ji} are symmetrical and the adjoint operator of $\tilde{\Delta}$ is defined by

$$\tilde{\Delta}*v=\sum_{i,j}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial v}{\partial x_i}\right),$$

then $\tilde{\Delta}=\tilde{\Delta}*$, see [3].

Now, we define the operator Pi in the same way as in [3]. In order to this article be as self-contained as possible, we reproduce some calculations shown in [3].

Let Pi be the first order differential operator defined by:

$$P_i[u,v]=\left(\sum_j a_{i,j}(x)\frac{\partial u}{\partial x_j}\right)v-u\sum_j a_{j,i}(x)\frac{\partial v}{\partial x_j}, \tag{2.2}$$

where u,v are twice continuously differentiable functions in $\bar{\Omega}$ with values in $\mathcal{A}_n(2,\alpha_i,\gamma_{ij})$.Then we have

$$\sum_{i=0}^{n}\frac{\partial P_i[u,v]}{\partial x_i}=(\tilde{\Delta}u)v-u\tilde{\Delta}v. \tag{2.3}$$

On the other side, using that $a_{ji}=a_{ij}$, we have

$$\sum_{j=1}^{n}P_i[u,v]N_i=|\tilde{N}|\left[\left(\frac{\partial u}{\partial \tilde{N}}\right)v-u\frac{\partial v}{\partial \tilde{N}}\right], \tag{2.4}$$

where $Ni, i=0,\cdots n$, are the components of the unit outer normal vector to $\partial\Omega$ and Ñ is the co-normal vector with respect to $\tilde{\Delta}$, i.e., the unit vector with the direction of the vector

$$\hat{N}=\left(\sum_i a_{i0}N_i,\sum_i a_{i1}N_i,\dots,\sum_i a_{in}N_i,\right).$$

The **Gauss Integral Formula** states

$$\int_\Omega\sum_{i=0}^{n}\frac{\partial f_i}{\partial x_i}dx=\int_{\partial\Omega}\sum_{i=0}^{n}f_iN_i\,d\mu\,,$$

Provide the boundary $\partial\Omega$ is sufficiently smooth and the function f_i are continuously differentiable in $\bar{\Omega}$. Applying this formula to the functions Pi defined by (2.2) and taking into account the relations (2.3) and (2.4), one gets the following **Green type Integral Formula** for the operator $\tilde{\Delta}$:

$$\int_\Omega\left((\tilde{\Delta}u)v-u\tilde{\Delta}v\right)dx=|\tilde{N}|\int_{\partial\Omega}\left(\frac{\partial u}{\partial\tilde{N}}v-u\frac{\partial v}{\partial\tilde{N}}\right)d\mu \tag{2.5}$$

3 INTEGRAL FORMULAS

3.1 THE NON-EUCLIDEAN DISTANCE

For the equation (2.1) equation, we define the quadratic form:

$$\begin{pmatrix}1&0&0&\dots&0\\0&\alpha_1&-\gamma_{12}&\dots&-\gamma_{1n}\\0&-\gamma_{21}&\alpha_2&\dots&-\gamma_{2n}\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&-\gamma_{n1}&-\gamma_{n2}&\dots&\alpha_n\end{pmatrix}. \tag{3.1}$$

When the aj are positive, one finds that equation (2.1) is elliptic, provided the absolute values of the γ_{ij} are not too large. In other words, if $\alpha_j>0$ and $|\gamma_{ij}|\leq$ const, with a suitable constant, then (2.1) is elliptic.

As consequence of the ellipticity. Then (3.1) has an inverse matrix having the form

$$\begin{pmatrix}1&0&\dots&0\\0&A_{11}&\dots&A_{1n}\\\vdots&\vdots&\ddots&\vdots\\0&A_{n1}&\dots&A_{nn}\end{pmatrix}, \tag{3.2}$$

where $A_{ij}=A_{ji}$ (because of $\gamma_{ij}=\gamma_{ji}$). Using these coefficients, define for two points $\xi=$

$(\xi_0, \xi_1, \dots, \xi_n)$ and $x = (x_0, x_1, \dots, x_n)$ of $\mathbb{R}^{n+1}$ a (non-Eulidean) distance ρ by

$$\rho^2 = (x_0 - \xi_0)^2 + \sum_{i,j=1}^{n} A_{ij}(x_i - \xi_i)(x_j - \xi_j). \quad (3.3)$$

3.2 THE PARAMETER NEWTON KERNEL

Now we are in position to consider the function defined in [3] by

$$\widetilde{K}(x,\xi) = \frac{-1}{(n-1)w_{n+1}} \cdot \frac{1}{\rho^{n-1}},$$

where w_{n+1} the surface measure of the units phere and ρ is the distance given by (3.3). In [3] was proved that the function $\widetilde{K}(x,\xi)$ is a weakly singular function at ξ and solution of the equation $\tilde{\Delta} u = 0$, for $x \neq \xi, x, \xi \in \Omega$. See [18]

Applying (2.5) with $u = \widetilde{K}(x,\xi)$ on the domain $\Omega_\varepsilon = \Omega \setminus U_\varepsilon(\xi)$, where $U_\varepsilon(\xi)$ is the ε-neighborhood of ξ, we obtain

$$-\int_{\Omega\varepsilon} \widetilde{K} \cdot \tilde{\Delta} v \, dx = |\widetilde{N}| \int_{\partial\Omega} \left(\frac{\partial \widetilde{K}}{\partial \widetilde{N}} v - \widetilde{K} \frac{\partial v}{\partial \widetilde{N}}\right) d\mu - |\widetilde{N}| \int_{|x-\xi|=\varepsilon} \left(\frac{\partial \widetilde{K}}{\partial \widetilde{N}} v - \widetilde{K} \frac{\partial v}{\partial \widetilde{N}}\right) d\mu. \quad (3.5)$$

Taking into account that v is a twice continuously differentiable function in $\overline{\Omega}$, the condition that $\widetilde{K}$ is weakly singular at ξ and the Schmidt inequality [21], we get

$$\lim_{\varepsilon \to 0} \int_{\Omega\varepsilon} \widetilde{K} \cdot \tilde{\Delta} v \, dx = \int_{\Omega} \widetilde{K} \cdot \tilde{\Delta} v \, dx$$

To calculate

$$\lim_{\varepsilon \to 0} \int_{|x-\xi|=\varepsilon} \sum_{i=0}^{n} P_i[K, v] \, N_i d\mu,$$

we observe from (2.4) that

$$\sum_{i=0}^{n} P_i[K, v] \, N_i = |\widetilde{N}| \left(\frac{\partial \widetilde{K}(x,\xi)}{\partial \widetilde{N}} (v(x) - v(\xi)) + \frac{\partial \widetilde{K}(x,\xi)}{\partial \widetilde{N}} v(\xi) - \widetilde{K}(x,\xi) \frac{\partial v}{\partial \widetilde{N}} \right) \quad (3.6)$$

Using (3.6) and the same arguments as in [3] pages (532 - 533), we obtain

$$\lim_{\varepsilon \to 0} \int_{|x-\xi|=\varepsilon} \sum_{i=0}^{n} P_i[K, v] \, N_i d\mu = c(\alpha_i, \gamma_{ij}) \cdot v(\xi) \cdot \quad (3.7)$$

We recall that as the function $\widetilde{K}(x,\xi)$ is a weakly singular function at ξ and solution of the equation $\tilde{\Delta} u = 0$, for $x \neq \xi, x, \xi \in \Omega$, the limit (3.7) shows that $\widetilde{K}(x,\xi)$ is a fundamental solution for the operator $\tilde{\Delta}$. We call these function the parameter Newton kernel in $\mathscr{A}_2(2, \alpha_j, \gamma_{ij})$

Finally, carrying out the limiting process $\varepsilon \to 0$ in (3.5), we get the **Cauchy-Pompeiu type Integral** Representation Formula.

Theorem 3.1. *Suppose v is a twice continuously differentiable function in* $\overline{\Omega}$ *with values in the Clifford type algebra* $\mathscr{A}_2(2, \alpha_j, \gamma_{ij})$, *over orientable manifold with boundary* $\partial\Omega$. *Then we have*

$$c(\alpha_j, \gamma_{ij}) \cdot v(\xi) = |\widetilde{N}| \int_{\partial\Omega} \left(\frac{\partial \widetilde{K}}{\partial \widetilde{N}} v - \widetilde{K} \frac{\partial v}{\partial \widetilde{N}}\right) d\mu + \int_{\Omega} \widetilde{K} \cdot \tilde{\Delta} v \, dx \quad (3.8)$$

for points ξ *in* Ω.

If the function v is a solution of equation (2.1), then Theorem 3.1 leads also to the following representation by boundary integrals:

Corollary 3.2. *Let v be an* $\mathscr{A}_2(2, \alpha_j, \gamma_{ij})$*-valued solution of equation* $\tilde{\Delta} v = 0$, *then*

$$c(\alpha_j, \gamma_{ij}) \cdot v(\xi) = |\widetilde{N}| \int_{\partial\Omega} \left(\frac{\partial \widetilde{K}}{\partial \widetilde{N}} v - \widetilde{K} \frac{\partial v}{\partial \widetilde{N}}\right) d\mu \quad (3.9)$$

for points ξ *in* Ω.

Remark 3.3. The Cauchy-Pompeiu type formula (3.8) is different to that obtained in the paper [3]. Due to the non-commutativity of the algebra (see (1.3)). Other Cauchy-Pompeiu type integral formula can be obtained considering $v = \widetilde{K}(x,\xi)$ in (2.5).

Theorem 3.4. *Suppose u is a twice continuously differentiable function in* $\overline{\Omega}$ *with values in the Clifford type algebra* $\mathscr{A}_2(2, \alpha_j, \gamma_{ij})$. *Then we have*

$$u(\xi) \cdot c(\alpha_j, \gamma_{ij}) = -|\widetilde{N}| \int_{\partial\Omega} \left(\frac{\partial u}{\partial \widetilde{N}} \widetilde{K} - u \frac{\partial \widetilde{K}}{\partial \widetilde{N}}\right) d\mu + \int_{\Omega} \tilde{\Delta} u \cdot \widetilde{K} \, dx \quad (3.10)$$

for points ξ *in* Ω

3.3 A DISTRIBUTIONAL SOLUTION

Using the parameter Newton kernel $\widetilde{K}(x,\xi)$ we will construct a distributional solution for the inhomogeneous equation $\tilde{\Delta} u = -h$. We denote Ω_x as the domain W with respect to the x −space and Ω_ξ as the domain Ω with respect to the ξ −space.

Theorem 3.5. *Let h be an integrable function with values in* $\mathscr{A}_2(2, \alpha_j, \gamma_{ij})$. *Then the function u defined by* *is a distributional solution of the inhomogeneous equation* $\Delta u = -h$, *where* $c(\alpha_i, \gamma_{ij})$ *is a Clifford number in* $\mathscr{A}_2(2, \alpha_j, \gamma_{ij})$ *with inverse.*

Proof. Suppose j is a test function with values in $\mathscr{A}_2(2, \alpha_j, \gamma_{ij})$. Replacing u by φ in the Cauchy-Pompeiu formula (3.8), we obtain

$$\varphi(\xi) = c^{-1}(\alpha_i, \gamma_{ij}) \cdot \int_\Omega \widetilde{K} \cdot \tilde{\Delta}\varphi \, dx.$$

Taking into account the Fubini's theorem for weakly singular integrands, we have

$$\int_{\Omega x} u \cdot \tilde{\Delta}\varphi \, dx = \int_{\Omega x} \left(\int_{\Omega\xi} (-h)(\xi) \cdot c^{-1}\left(\alpha_i, \gamma_{ij}\right) \cdot \widetilde{K}(x,\xi) d\xi \right) \tilde{\Delta}\, \varphi \, dx = \int_{\Omega\xi} (-h)(\xi) \cdot c^{-1}\left(\alpha_i, \gamma_{ij}\right) \cdot \left(\int_{\Omega x} \widetilde{K}(x,\xi) \cdot \tilde{\Delta}\, \varphi \, dx \right) d\xi = \int_{\Omega\xi} (-h)(\xi) \cdot \varphi(\xi) d\xi.$$

Using the Cauchy-Pompeiu type formula (3.10) the following theorem can be proven similarly

Theorem 3.6. Let h *be an integrable function with values in* $\mathscr{A}_2(2, \alpha_j, \gamma_{ij})$. *Then the function u defined by*

$$u(x) = \int_\Omega \widetilde{K}(x,\xi) \cdot c^{-1}\left(\alpha_i, \gamma_{ij}\right) \cdot (-h)(\xi) d\xi$$

is a distributional solution of the inhomogeneous equation $\Delta u = -h$, where $c(\alpha_i, \gamma_{ij})$ is a Clifford number in $\mathscr{A}_2(2, \alpha_j, \gamma_{ij})$. with inverse.

Remark 3.7. Note that the Cauchy-Pompeiu formula showing in the paper [3] is not useful in order to obtain distributional solutions of the differential equations $\Delta u = -h$.

3.4 THE GENERAL SOLUTION OF THE INHOMOGENEOUS EQUATION $\tilde{\Delta} u = -h$.

Theorem 3.8. *Let h be a continuous function in the algebra generated by (1.3) then The general solution of the inhomogeneous equation* $\tilde{\Delta} u = -h$. *given by*

$$u(x) = u_h + \int_\Omega (-h)(\xi) \cdot c^{-1}\left(\alpha_i, \gamma_{ij}\right) \cdot \widetilde{K}(x,\xi) d\xi,$$

where u_h *is an arbitrary solution of the homogeneous equation* $\tilde{\Delta} u = 0$ *and c is a Clifford number.*

Proof. It is clear that $\tilde{\Delta}\left(u_h + \int_\Omega (-h)(\xi) \cdot c^{-1}\left(\alpha_i, \gamma_{ij}\right) \widetilde{K}(x,\xi) d\xi,\right)$. On the other hand, let u any distributional solution of the equation $\tilde{\Delta} u = -h$. Then the difference of both functions

$$u_h = u(x) - \int_\Omega (-h)(\xi) \cdot c^{-1}\left(\alpha_i, \gamma_{ij}\right) \widetilde{K}(x,\xi) d\xi$$

is a distributional solution of the equation $\tilde{\Delta} u = 0$. 0Then by Weyl's Lemma [17] for the elliptic equation $\tilde{\Delta} u = 0$ we get that u_h must be a classical solution of this equation.

4 APPLICATIONS: COMBINATIONS OF FIRST AND SECOND ORDER OPERATORS

In this section we consider the power two of the operator $\tilde{\Delta}$ and we use the Cauchy-Pompeiu formula in order to solve $\tilde{\Delta}^2 u = h$ where h is continuous. Also we consider the mixed of the operator, for instance $D\tilde{\Delta} u = -h.$, where h is also continuous.
In order to show the combinations we need the integral representation formula of Cauchy-Pompeiu type for the operator of order one $D - \lambda$, where λ is constant, the name of this operator is the metamonogenic operator of order one. [5].

4.1 CAUCHY-POMPEIU TYPE FORMULA FOR $D - \lambda = D_\lambda$

Using this distance ρ define in (3.3), we can define the kernel for D as

$$E(x,\xi) = \frac{1}{\omega_{n+1}} \frac{1}{\rho^{n+1}} \left((x_0 - \xi_0) - \sum_{i,k=1}^{n} e_i A_{ik} (x_k - \xi_k) \right) \tag{4.1}$$

See [4, 11] for the proof.

If

$$\wp_\lambda(x,\xi) = \lambda(x_0 - \xi_0) \tag{4.2}$$

for $\lambda \epsilon \mathbb{R}$, we can define

$$E_\lambda(x,\xi) = E(x,\xi) e^{\wp_\lambda(x,\xi)}. \tag{4.3}$$

Theorem 4.1. The function $E_\lambda(x,\xi)$ is a fundamental solution for the operator D_λ, for $x \neq \xi$. For the proof of this theorem, see [4, 11].

Finally, as part of the proof of the previous theorem, one consequence is that we get the **Cauchy-Pompeiu type** integral formula for D_λ operator.

Theorem 4.2. *Let u be a twice continuously differentiable function with values in* $\overline{\Omega} \subset$

$\mathscr{A}^*_{n,2}$. *Then, at each interior point ξ of Ω we have*

$$c\left(\alpha_j,\gamma_{ij}\right)\cdot u(\xi) = \int_{\partial\Omega} E_{-\lambda}(x,\xi)\cdot d\sigma\cdot u - \int_{\Omega} E_{-\lambda}(x,\xi)\cdot D_\lambda u dx. \quad (4.4)$$

4.2 INTEGRAL REPRESENTATION FOR THE EQUATION $\tilde{\Delta}^2 u = h$ WITH RIGHT HAND SIDE CONTINUOUS

Let W be a bounded domain in Rn+1 with sufficiently smooth boundary. Consider the following homogeneous equation:

$$\tilde{\Delta}^2\omega = \tilde{\Delta}\tilde{\Delta}\omega = h, \quad (4.5)$$

where ω is a four times continuously differentiable function in $\overline{\Omega}$.

Following the ideas given in [6, 4], this equation is decomposed into the system

$$\tilde{\Delta}\omega = \Phi \quad (4.6)$$
$$\tilde{\Delta}\Phi = h \quad (4.7)$$

Using (3.8) and (4.6), we have the following integral representation for ω:

$$c(\alpha_j,\gamma_{ij})\cdot\omega(\xi) = |\tilde{N}|\int_{\partial\Omega}\left(\frac{\partial\tilde{K}}{\partial\tilde{N}}\omega - \tilde{K}\frac{\partial\omega}{\partial\tilde{N}}\right)d\mu + \int_\Omega \tilde{K}\,\Phi\, dx, \quad (4.8)$$

for points ξ in Ω. On the other hand, using (3.9) and (4.7) we also get an integral representation for Φ, if $c(\alpha_j,\gamma_{ij})$ admits inverse, then we have:

$$\Phi(\xi) = \int_\Omega \tilde{L}(x,\xi)\cdot c_1^{-1}\left(\alpha_i,\gamma_{ij}\right)\cdot h(\xi)d\xi \quad (4.9)$$

Substituting (4.9) in (4.8), we rewrite the integral representation for $\tilde{\Delta}^2 u = h$ as

$$c(\alpha_j,\gamma_{ij})\cdot\omega(\xi) = |\tilde{N}|\int_{\partial\Omega_x}\left(\frac{\partial\tilde{K}(x,\xi)}{\partial\tilde{N}}\omega(x) - \tilde{K}(x,\xi)\frac{\partial\omega(x)}{\partial\tilde{N}}\right)d\mu + \int_{\Omega_x}\int_{\Omega_{\tilde{\xi}}}\tilde{K}(x,\xi)\left(\tilde{K}(\xi,\xi)\cdot c_1^{-1}\left(\alpha_i,\gamma_{ij}\right)\cdot h(\tilde{\xi})\right) d\tilde{\xi}\, dx, \quad (4.10)$$

for points ξ in Ω.

Here, Ω_x means the domains Ω respect to the variable x and $\Omega_{\tilde{\xi}}$ is the domain Ω respect to the variable $\tilde{\xi}$; while $d\Omega_x$ denote the boundary of Ω_x and $d\Omega_{\tilde{\xi}}$ denote the boundary of $\Omega_{\tilde{\xi}}$

4.3 INTEGRAL REPRESENTATION FOR $\tilde{\Delta}^2(D-\lambda)w = 0$

In this section, we present an example which illustrates the use of the formulas set forth in the preceding sections. Using the Cauchy-Pompeiu Integral Formula (4.4) and the previous result (4.10),
we can obtain an integral representation for $\tilde{\Delta}^2(D-\lambda)w = 0$ as follows:

Consider the system

$$\tilde{\Delta}^2\omega = \Phi \quad (4.11)$$
$$\Phi(D-\lambda) = h \quad (4.12)$$

Using the formula (4.10), we have for the functions ω in (4.17) the following representation

$$c(\alpha_j,\gamma_{ij})\cdot\omega(\xi) = |\tilde{N}|\int_{\partial\Omega_x}\left(\frac{\partial\tilde{K}(x,\xi)}{\partial\tilde{N}}\omega(x) - \tilde{K}(x,\xi)\frac{\partial\omega(x)}{\partial\tilde{N}}\right)d\mu + \int_{\Omega_x}\int_{\Omega_{\tilde{\xi}}}\tilde{K}(x,\xi)\left(\tilde{K}(\xi,\xi)\cdot c_1^{-1}\left(\alpha_i,\gamma_{ij}\right)\cdot\Phi\right) d\tilde{\xi}\, dx, \quad (4.13)$$

On the other hand for (4.12) we use the Cauchy-Pompeiu Integral Formula (4.4) assuming that $c_2(\alpha_j,\gamma_{ij})$ has an inverse, we obtain:

$$\Phi(\xi) = c_2^{-1}(\alpha_j,\gamma_{ij})\cdot\int_{\partial\Omega}E_{-\lambda}(x,\xi)\cdot d\sigma\cdot\Phi(x) - c_2^{-1}(\alpha_j,\gamma_{ij})\cdot\int_\Omega E_{-\lambda}(x,\xi)\cdot D_\lambda\cdot\Phi(x)\,dx \quad (4.14)$$

or

$$\Phi(\xi) = c_2^{-1}(\alpha_j,\gamma_{ij})\cdot\int_{\partial\Omega_\xi}E_{-\lambda}\left(\tilde{\xi},\tilde{\tilde{\xi}}\right)\cdot d\sigma\cdot\tilde{\Delta}^2(\tilde{\xi}) - c_2^{-1}(\alpha_j,\gamma_{ij})\cdot\int_{\Omega_\xi}E_{-\lambda}\left(\tilde{\xi},\tilde{\tilde{\xi}}\right)\cdot D_\lambda\left(\tilde{\Delta}^2(\tilde{\xi})\right)dx \quad (4.15)$$

Substituting the above expression in (4.10), we obtain the following integral representation for the function ω:

$$c(\alpha_j,\gamma_{ij})\cdot\omega(\xi) = |\tilde{N}|\int_{\partial\Omega_x}\left(\frac{\partial\tilde{K}(x,\xi)}{\partial\tilde{N}}\omega(x) - \tilde{K}(x,\xi)\frac{\partial\omega(x)}{\partial\tilde{N}}\right)d\mu + \int_{\Omega_x}\int_{\Omega_{\tilde{\xi}}}\tilde{K}(x,\xi)(\tilde{K}(\xi,\xi)\cdot c_1^{-1}\left(\alpha_i,\gamma_{ij}\right)\cdot c_2^{-1}(\alpha_j,\gamma_{ij})\cdot\int_{\partial\Omega_\xi}E_{-\lambda}\left(\tilde{\xi},\tilde{\tilde{\xi}}\right)\cdot d\sigma\cdot\tilde{\Delta}^2(\tilde{\xi}))\, d\tilde{\xi}dx +$$

$$\int_{\Omega_x}\int_{\Omega_{\tilde{\xi}}} \tilde{K}(x,\xi)(\tilde{K}(\xi,\xi)\cdot c_1^{-1}\left(\alpha_i,\gamma_{ij}\right)\cdot$$
$$c_2^{-1}(\alpha_j,\gamma_{ij})\cdot\int_{\partial\Omega_\xi} E_{-\lambda}\left(\tilde{\xi},\tilde{\tilde{\xi}}\right)\cdot D_\lambda\cdot\left(\tilde{\Delta}^2(\tilde{\xi})\right))\,d\tilde{\xi}dx$$

4.4 INTEGRAL REPRESENTATION FORMULA FOR THE EQUATION $\tilde{\Delta}(D-\lambda)\omega = h$ WITH RIGHT HAND SIDE CONTINUOUS

Now we are in position to give an integral representation for solutions of the following equation:

$$\tilde{\Delta}(D-\lambda)\omega = h, \qquad (4.16)$$

for a three times continuously differentiable function ω in $\overline{\Omega}$. This problem is equivalent to find an integral representation for the solutions of the inhomogeneous equation $(D-\lambda)\omega = \Phi$, where the right-hand function F is a solution of the equation $\tilde{\Delta}\Phi = h$.
Considering the system

$$(D-\lambda)\omega = \Phi, \qquad (4.17)$$
$$\tilde{\Delta}\Phi = h \qquad (4.18)$$

Then we have the following integral representation for the functions ω in (4.17)

$$\mathrm{c}(\alpha_j,\gamma_{ij})\cdot\omega(\xi) = \int_{\partial\Omega} E_{-\lambda}(x,\xi)\cdot d\sigma\cdot\omega(x) - \int_{\Omega} E_{-\lambda}(x,\xi)\cdot\Phi(x)dx \qquad (4.19)$$

On the other hand, for the equations (4.18) using the distributional solution, we obtain the following representation

$$\Phi(\xi) = \int_{\partial\Omega_\xi}\tilde{K}\left(\tilde{\xi},\tilde{\tilde{\xi}}\right)\cdot c_1^{-1}\left(\alpha_i,\gamma_{ij}\right)\cdot h(\xi)d\xi \quad (4.20)$$

Summarizing,

$$\mathrm{c}(\alpha_j,\gamma_{ij})\cdot\omega(\xi) = \int_{\partial\Omega} E_{-\lambda}(x,\xi)\cdot d\sigma\cdot\omega(x) - \int_{\partial\Omega_\xi} E_{-\lambda}\left(x,\tilde{\tilde{\xi}}\right)\cdot\int_{\Omega} E_{-\lambda}\left(\xi,\tilde{\xi}\right)\cdot c_1^{-1}(\alpha_i,\gamma_{ij})\cdot h(\tilde{\xi})d\tilde{\xi}\cdot dx. \qquad (4.21)$$

5 BRIEFLY DISCUSSION ON THE POSSIBLE PHYSICAL PROPERTIES

In this section we discuss one interesting physical example where we can argue on the implications of our results concerning a currently hot topic in condensed matter physics. Within this context, the study of the low energy physics of new materials, such as graphene, silicene and topological insulators leads to the analysis of the transport properties of Dirac fermions, i.e., particles that satisfy the standard Dirac equation (For a general discussion on the physical properties of topological insulators see reference [22]. Restricting to the two-dimensional case, the two dimensional Dirac Hamiltonian reads

$$\mathscr{H} = v_F(k_x\sigma_x + k_y\sigma_y) + m\sigma_z \qquad (5.1)$$

where $v_F \approx 10^5 m/s$ is called the Fermi velocity of the Dirac fermions whereas m is a (Real) tunable parameter called the effective mass and might depend on nature of the physical entities under consideration. In addition, si $\sigma_i = (i = x,y,z)$ are Pauli matrices describing spin degrees of freedom, whereas $\vec{k} = (k_x,k_y)$ is the momentum measured from the Dirac point.

The previously defined Pauli matrices satisfy the algebra

$$\sigma_i\sigma_j - \sigma_j\sigma_i = 2i\varepsilon_{ijk}\sigma_k \qquad (5.2)$$
$$\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij} \qquad (5.3)$$

with ε_{ijk} The totally anti symmetric Levi Civita symbol and di j the Kronecker delta.
Now, for a quasi-one dimensional system, we can focus on the motion along the x axis. Then we could use our results to describe scattering processes as described in the figure. In panel (a), we depict a typical scattering set up where an incident way (upper arrow) impinges on a potential barrier (in this case, of with L and amplitude V_0). Then, the quantum mechanical preservation of probability implies the outcome to be composed of two counter-propagating waves with amplitudes r (reflected) and t (for transmitted), which are in general complex numbers that must satisfy $|r|^2 + |t|^2 = 1$. This is schematically shown in the upper panel (a) of the figure. We could then used our approach to propose a scattering setup for modified Dirac fermions according to the lower panel (b) in the figure. Here, a left incoming (generally bicomplex) propagating wave would be also scattered off a potential barrier. Yet, the amplitudes now being T and R could but must

not longer be defined as complex parameters, since, accordingly to our integral representation for the modified Laplace $\widetilde{\Delta}$ operator, one could find solutions with a more general structure.
In particular, the expansion coefficients k_j that appear in the solutions depicted in panel (b) could be interpreted as those extensions of the standard quantum momentum eigenvalues in the momentum space representation of the quantum mechanical wave function. One important point here is the nature of the boundary conditions that lead to a well-defined solution.

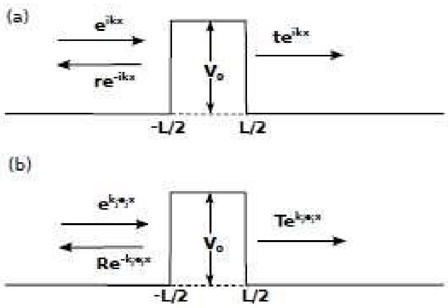

Figure 1: *Quasi-one dimensional system*

We would expect our results to be relevant for exotic quasi particles with much more complex internal physical structure as it happens in the context of the anyonic particles, i.e. those quantum particles whose composite Hilbert space leads to more general statistical properties as compared to Dirac-Fermi or Bose-Einstein statistics. It has been already established in the literature that the anyonic wavefunctions in a two-dimensional space are just 1-dimensional representations of the braid group [23, 24, 25, 26]. This is in turn related to the fractional quantum Hall effect and we expect to provide new Clifford algebraic representations of such extensions of the underlying Lie algebra.
Needles to say, one could also resort to numerical solutions to deal with more complex situations. Yet, this approach goes beyond the goals of the present work and could be the subject of future work.

REFERENCIAS BIBLIOGRAFICAS Y ELECTRONICAS

[1] **T. Marimoto and A. Furusaki** *Topological classification with additional symmetries from Clifford algebras*. Phys. Rev. B 88, 125129, (2013).

[2] **B-Y Yang and N**. *Nagaosa Classification of stable three-dimensional Dirac semimetals with nontrivial topology*. Nature Communications 5, 4898, (2014).

[3] **E. Ariza, A. Di Teodoro A, A**. Infante, and J. Vanegas, *Fundamental Solutions for Second Order Elliptic Operators in Clifford-Type Algebras*. Adv. Appl. Clifford Algebras 25, 527-538, (2015).

[4] **C. Balderrama, A. Di Teodoro A, and A.** Infante, *Some integral representation for metamonogenic functions in Clifford algebras depending on parameters.* Adv. Appl. Clifford Algebras 23, 793-813, (2013).

[5] **M. Balk**. *Polyanalytic functions, Berlin: Akademie Verlag*, (1991).

[6] **H. Begehr**, *Boundary value problems in complex analysis I and II. Boletín de la Asociación Matemática Venezolana*, Vol. XII, Nro 1-2, 65-85, 217-250, (2005).

[7] **H. Begehr**, *Integral representations in complex, hypercomplex and Clifford analysis. Integral transform and Special Functions* 13, 223-241, (2002).

[8] **H. Begehr, R. Gilbert, Wen-Chung Guo.** *Partial Differential and Integral Equations (International Society for Analysis, Applications and Computation, International Society for Analysis, Applications and Computation (Book 2)*. Springer; Softcover reprint of the original (1999) edition.