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UNIFORMLY CONTINUOS SUPERPOSITION OPERATORS ON SPACES OF FUNCTIONS OF BOUNDED VARIATION DEFINED ON COMPACT SUBSET OF C

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Resumen: En este artículo mostramos que si la función generadora h de un operador de superposición H, es continua en la primera variable y si H envía un subconjunto del BV(s), el espacio de las funciones de variación acotada sobre subconjuntos compactos de C (el plano complejo), en otro espacio específico entonces la función generadora h es afín en la variable funcional.
Palabras claves: Operador de superposición, variación de una función, uniformemente continuos, subconjuntos compactos.

Abstract: In this paper we show that if the generating function h, of a uniformly continuous superposition operator H, is continuous in the first variable and if H sends a range-restricted subset of BV (σ), the space of functions of bounded variation on compact subset $\sigma \subset C$, into another such space, then the function h must be affine in the functional (second) variable.
Keywords: Superposition operator, Variation of a function, uniformly continuous, Compact subsets.
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1. INTRODUCTION

Let σ, M and N be arbitrary non-empty subsets. Denote by Mσ the set of all functions from σ to M. Given a function h: $\sigma \times N \to M$, the map H: Nσ → Mσ defined by Hf (x):= h(x, f (x)) for all $x \in \sigma$ and $f \in N\sigma$, is called the superposition (or Nemytskij) operator generated by h.

This operator plays an important role in various mathematical fields, e.g. in the theory of nonlinear integral equations, and has been studied thoroughly. Perhaps, the most important problem concerning the theory of superposition operators, is to establish necessary and sufficient conditions under which such operator maps a given function space into itself. These conditions are called acting conditions (e.g., (non-linear) boundedness, continuity, local or global Lipschitz conditions, etc.). On the other hand, being superposition operators the simplest operators between function spaces, another important problem, is to determine if a certain given operator, that acts between some given function spaces, can be redefined via the notion of superposition, thus, e.g., it has been established that for some function spaces, any locally defined operator is a Nemitskij operator (cf. [8], [9] and [6]). We refer the reader to [1] in which most of the basic facts and results concerning superposition operators are exposed.

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2. NOTATION AND BASIC DEFINITIONS

In the first place we present the definition of variation throughout a curve as it was introduced by Ashton in [2] and then we present the definition and main properties of the notion of bounded variation for complex valued functions defined on a compact subset σ of C (see [4]).

Throughout this section σ denotes a non-empty compact subset of C.

A curve, or path, in C is a continuous function $\gamma : [0, 1] \to C$. The length of a curve γ, denoted by $\ell(\gamma)$, is the supremun of the lengths of all the polygonal that can be inscribed in the curve (that is: whose vertices lie on γ).

As usual we will denote by $\gamma_1 + \gamma_2$ the juxtaposition of the paths γ_1 and γ_2 such that $\gamma_1(1) = \gamma_2(0)$; that is

$$(\gamma_1 + \gamma_2)(t) := \begin{cases} \gamma_1(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ \gamma_2(2t-1) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

Definition 2.1. Let γ be a path. We will say that $\{z_i\}_{i=1}^n$ is a partition of γ over σ if $z_i \in \sigma$ for all i and if there is a partition $\{s_i\}_{i=1}^n \in \Lambda([0,1])$ such that $z_i = \gamma(s_i)$ for all *i*. The set of all partitions of γ over σ will be denoted by Λ(γ, σ).

Definition 2.2 ([2]). Let σ be a compact subset of C, f: σ → C and let γ be a path in σ. The variation of f throughout the path γ is defined as:

$$cVar(f,\gamma,\sigma) = cVar(f,\gamma) := \sup_{\{z_j\}_{j=1}^n \in \Lambda(\gamma,\sigma)} \sum_{j=1}^{n-1} |f(z_{j+1}) - f(z_j)|.$$

In the following proposition we list some basic properties of the functionals of c-variation. A proof of these properties can be found in [2].

Proposition 2.3([2]). Let σ1 ⊆ σ2 be non-empty, with σ1 non a singleton, compact subsets of C, f, g : σ2 → C, γ ∈ C([0, 1]) and *k* ∈ C. Then

(i) $cVar(f+g,\gamma) \le cVar(f,\gamma) + cVar(g,\gamma)$.

(ii) *For* $\gamma \in C([0,1])$, *if* $cVar(f,\gamma) < \infty$ *then* f *is bounded over the set* $\{z \in \mathbb{C} : z = \gamma(t)$ *for some* $t \in [0,1]\}$.

(iii) $cVar(fg,\gamma) \le \|f\|_\infty cVar(g,\gamma) + \|g\|_\infty cVar(f,\gamma)$.

(iv) $cVar(kf,\gamma) = |k| cVar(f,\gamma)$.

(v) *If* $\gamma = \gamma_1 + \gamma_2$ *then*
   (a) $cVar(f,\gamma) = cVar(f,\gamma_1) + cVar(f,\gamma_2)$ *and* $cVar(f,\gamma_1) \le cVar(f,\gamma)$.
   (b) $cVar(f,\gamma,\sigma_1) \le cVar(f,\gamma,\sigma_2)$.
   (c) *Let* $f : \sigma \to \mathbb{C}$, $\gamma_1, \gamma_2 \in C([0,1])$ *and suppose that* $\gamma_1 \cong \gamma_2$. *Then* $cVar(f,\gamma_1) = cVar(f,\gamma_2)$.

Recall that a polygonal (path) is a curve γ : [0, 1] → C for which there is a partition t0 < t1 < t2 < · · · < tn such that γ(t) is linear on each subinterval [tk, tk+1]. The range of γ, when t runs through [tk, tk+1], is called a side of the

poligonal γ and it is denoted by [γ(tk), γ(tk+1)]. The set of all polygonal that meet σ will be denoted as (σ) or simply by □.

Definition 2.4([4]). Let f: σ → C. The variation of *f* on σ is defined as

$$Var(f,\sigma) := \sup_{\gamma\in\Gamma} \frac{cVar(f,\gamma)}{\ell(\gamma)}.$$

The set

$$BV(\sigma) = BV(\sigma,\mathbb{C}) := \{f : \sigma \to \mathbb{C} : Var(f,\sigma) < +\infty\}.$$

will be called the space of (complex) functions of bounded variation on σ.

Remark 2.5. Notice that if Range(γ) ∩ σ= ∅, then, without lost of generality, we may assume that both γ(0) and γ(1) are points of σ, since otherwise we can re-parameterize γ such that 0 is the smallest of the entrance points of γ over σ and 1 is the largest of the exit points of γ over σ.

A proof of the following Lemma can be found in [4].

Lemma 2.6([4]). If Var(f, σ) < +∞ then f is bounded in σ.

As a consequence of the previous lemma it follows that the function f(z) :=1/z is not of bounded variation on any compact set that contains zero.

Next we present some properties of the variation functionals. A proof of them may be found in [4].

Theorem 2.7([4]). Let σ be a non-empty compact subset of C, f, g : σ → C and k ∈ C. Then

(i) $Var(f,\sigma) = 0$ *if and only if* f *is a constant function.*
(ii) $Var(f+g,\sigma) \le Var(f,\sigma) + Var(g,\sigma)$.
(iii) $Var(kf,\sigma) = |k| Var(f,\sigma)$.
(iv) $Var(fg,\sigma) \le \|f\|_\infty Var(g,\sigma) + \|g\|_\infty Var(f,\sigma)$.

Theorem 2.7 guarantees that BV (σ) is a linear space. Next, we define the functional

$$\|f\|_{BV(\sigma)} := \|f\|_\infty + Var(f,\sigma), \tag{2.1}$$

On *BV(σ)*, where, ||·||∞ is the well-known sup-norm. Also, from Theorem 2.7 it follows that $\|\cdot\|_{BV(\sigma)}$ defines a norm on BV(σ) and it can be shown that, in fact, it is a Banach algebra with respect to this norm (see [4]).

3. MAIN RESULTS

To prove our main result, we will need to prove the following results.

Lemma 3.1. Let Ω⊆C be a convex set such that 0 ∈ Ω, and let f : Ω → C be a solution of equation.

$$f\left(\frac{z_1+z_2}{2}\right) = \frac{f(z_1)+f(z_2)}{2}, \tag{3.1}$$

Such that

$$f(0) = 0. \tag{3.2}$$

Then, for every z ∈ Ω and *n* ∈ N,

$$f\left(\frac{z}{2^n}\right) = \frac{1}{2^n} f(z). \tag{3.3}$$

Proof. Take an z ∈ Ω. Since is convex, $\frac{1}{2}(z+0) \in \Omega$, and by (3.1) and (3.2).

$$f\left(\frac{z}{2}\right) = f\left(\frac{z+0}{2}\right) = \frac{f(z)+f(0)}{2} = \frac{f(z)}{2}. \tag{3.4}$$

Thus (3.3) holds for n = 1. Assuming it true for an n ∈ N, we have

$$\frac{z}{2^{n+1}} = \frac{1}{2^{n+1}} z + \left(1 - \frac{1}{2^{n+1}}\right) 0 \in \Omega,$$

And by (3.3) for n and (3.4)

$$f\left(\frac{z}{2^{n+1}}\right) = \frac{1}{2^{n+1}}\left(\frac{z}{2}\right) = \frac{1}{2^{n+1}} f(z).$$

Induction completes the proof.

Lemma 3.2. Let ⊆ C be a convex set such that int 6= ∅, and let f: → C be a solution of equation (3.1). Fix an z0 ∈ int, and define the function f_0: − z_0 → C by

$$(3.5) \qquad f_0(z) = f(z_0 + z) - f(z_0).$$

Then there exists a unique function f_1: C → C satisfying equation (3.1) in C and such that

$$(3.6) \qquad f_1(z) = f_0(z) \quad for \quad x \in \Omega - z_0.$$

Proof. Function (3.5) is defined for z ∈ $\Omega - z_0$. Firs we verify that f0 satisfies equation (3.1) in $\Omega - z_0$. For every $z_1, z_2 \in \Omega - z_0$, we have $z_0 + z_1, z_0 + z_2 \in \Omega$, and by (3.5) and (3.1)

$$\begin{aligned} f_0\left(\frac{z_1+z_2}{2}\right) &= f\left(z_0 + \frac{z_1+z_2}{2}\right) - f(z_0) \\ &= f\left(\frac{z_0+z_1+z_0+z_2}{2}\right) - f(z_0) \\ &= \frac{1}{2}f(z_0+z_1) + \frac{1}{2}f(z_0+z_2) - f(z_0) \\ &= \frac{1}{2}[f(z_0+z_1) - f(z_0)] + \frac{1}{2}[f(z_0+z_2) - f(z_0)] \\ &= \frac{1}{2}[f_0(z_1) + f_0(z_2)]. \end{aligned}$$

Also, it is easily seen that 0 ∈ Ω − z0 and by (3.5)

$$(3.7) \qquad f_0(0) = 0$$

Now put 0 and n = 2n 0, n ∈ N. If x ∈ n, then z 2n ∈0. 0 is convex, just like, and 0 ∈ 0, whence z 2n+1 = 1 2 h z 2n + 0i ∈ 0, and z ∈ n+1. Thus,

$$(3.8) \qquad \Omega_n \subseteq \Omega_{n+1}, \qquad n \in \mathbb{N} \cup \{0\}.$$

Also, 0 ∈ int Ω_0, since z_0 ∈ int Ω. For every z ∈ C we have $\lim_{n\to\infty} \frac{z}{2^n} = 0$, whence it follows that there exists an $n \in \mathbb{N} \cup \{0\}$ such that $\frac{z}{2^n} \in \Omega_0$, whence z ∈ Ω_n. Hence

$$(3.9) \qquad \bigcup_{n=0}^{\infty} \Omega_n = \mathbb{C}.$$

Define the function f_1: C → C as follows:

$$(3.10) \qquad f_1(z) := 2^n f_0\left(\frac{z}{2^n}\right) \quad \text{if} \quad z \in \Omega_n, \quad n \in \mathbb{N} \cup \{0\}.$$

It is easy to check that whether definition (3.10) is correct. We must verify that it satisfies equation (3.1) in C. Take arbitrary z1, z2 ∈ C. By (3.9) and (3.8) there exists an $n \in \mathbb{N} \cup \{0\}$ such that $z_1, z_2 \in \Omega_n$.

Hence $\frac{z_1}{2^n}, \frac{z_2}{2^n} \in \Omega_0$, and $\frac{z_1+z_2}{2^{n+1}} = \frac{1}{2}\left(\frac{z_1}{2^n} + \frac{z_2}{2^n}\right) \in \Omega_0$, whence $\frac{z_1+z_2}{2^n} \in \Omega_n$. Now,

$$\begin{aligned} f_1\left(\frac{z_1+z_2}{2^n}\right) &= 2^n f_0\left(\frac{z_1+z_2}{2^{n+1}}\right) \\ &= 2^n\left[\frac{1}{2}f_0\left(\frac{z_1}{2^n}\right) + \frac{1}{2}f_0\left(\frac{z_2}{2^n}\right)\right] \\ &= \frac{f_1(z_1)+f_1(z_2)}{2}. \end{aligned}$$

Relation (3.6) results from (3.10) for n = 0.

To prove the uniqueness, suppose that a function f_2 : C → C satisfies equation (3.1) in C and fulfils the condition.

$$(3.11) \qquad f_2(z) = f_0(z) \quad \text{for} \quad z \in \Omega - z_0.$$

By (3.11) and (3.7) $f_2(0) = f_0(0) = 0$, and hence, by Lemma 3.1, $f_2\left(\frac{z}{2^n}\right) = \frac{1}{2^n} f_2(z)$, for z ∈ C, n ∈ N ∪ {0}. Take an arbitrary z ∈ C. By (3.9) there exists an n ∈ N∪{0} such that $z \in \Omega_n$, whence $\frac{z}{2^n} \in \Omega_0$. Thus we have by (3.11) and (3.10)

$$f_2(z) = 2^n f_2\left(\frac{z}{2^n}\right) = 2^n f_0\left(\frac{z}{2^n}\right) = f_1(z).$$

Consequently f2 = f1 in C.

Lemma 3.3. Let a function f: C → C satisfy equation (3.1) and relation (3.2). Then f is additive.
Proof. We have by Lemma 3.1 for arbitrary z_1, z_2 ∈ C.

$$f(z_1+z_2) = 2f\left(\frac{z_1+z_2}{2}\right) = 2\frac{f(z_1)+f(z_2)}{2} = f(z_1) + f(z_2),$$

i.e, f is additive.

Theorem 3.4. Let Ω ⊆ C be a convex set such that int(Ω) ≠∅, and let f: C→C be a solution of equation (3.1). Then there exist an additive unction g: C → C and constant a ∈ C such that

$$(3.12) \qquad f(z) = g(z) + a \quad for \quad z \in \Omega.$$

Proof. Fix an z_0 ∈ int and define the function f_0: $(\Omega - z_0)$ → C by (3.5). By Lemma 3.2 There exists a function f_1 : C → C satisfying equation (3.1) and condition (3.6). Hence by (3.7) $f_1(0) = f_0(0) = 0$. By Lemma 3.3 f_1 is additive. For arbitrary z ∈ we have z − z0 ∈ $\Omega - z_0$, whence y (3.5) and (3.6)

$$f(z) = f(z_0 + (z - z_0)) = f_0(z - z_0) + f(z_0) = f_1(z - z_0) + f(z_0).$$

Since f_1 is additive, we get hence

$$(3.13) \qquad f(z) = f_1(z) - f_1(z_0) + f(z_0).$$

Put $g = f_1$, $a = f(z_0) - f1(z0)$. Relation (3.12) results now from (3.13).

Remark 3.5. Given $\gamma \in C([0, 1])$ and $z1 = \gamma(t_1)$, $z_2 = \gamma(t_2)$ we will write $z_1 \leq z_2$ if $t_1 \leq t_2$. Analogously is defined $z_1 \geq z_2$.

Remark 3.6. Clearly, if $f \in BV(\sigma) \setminus \{0\}$ then $Var\left(\frac{f}{\|f\|_{BV(\sigma)}}, \sigma\right) \leq 1$.

The hard work is now accomplished, and we have everything we need to prove the main result.

Theorem 3.7. Let $\sigma \subseteq C$ be a compact subset, let $C \subseteq C$ be a convex set with non-empty interior and suppose that the generating function $h: \sigma \times C \to C$ of a superposition operator H, is continuous in the first variable. If H is uniformly continuous and if H sends the set $RC = \{f \in BV(\sigma) : f(\sigma) \subseteq C\}$ into $BV(\sigma)$ then there are functions $A,B : \sigma \to C$ such that

$$h(z, w) = A(z)w + B(z), \quad z \in \sigma \quad w \in C.$$

Moreover, if $0 \in C$ then $B \in BV(\sigma)$.

Proof. The proof will be divided in three steps:

Step 1. First of all we prove that if $f, g \in R_C$ then $|(H_f - H_g)(z) - (H_f - H_g)(\hat{z})|$ is bounded for all $\gamma \in$ □, and $z, \hat{z} \in \gamma([0, 1])$. Indeed, since H is uniformly continuous, its modulus of continuity operator $\omega : [0,+\infty] \to [0,+\infty]$ satisfies

$$\|\mathcal{H}f - \mathcal{H}g\|_{BV(\sigma)} \leq \omega(\|f - g\|_{BV(\sigma)}), \tag{3.14}$$

for all $f, g \in R_C$.

Hence, from (3.14) and Remark 3.6 we have

$$\begin{aligned} Var\left(\frac{\mathcal{H}f - \mathcal{H}g}{\omega(\|f-g\|_{BV(\sigma)})}, \sigma\right) &= Var\left(\frac{\mathcal{H}f - \mathcal{H}g}{\|\mathcal{H}f - \mathcal{H}g\|_{BV(\sigma)}} \frac{\|\mathcal{H}f - \mathcal{H}g\|_{BV(\sigma)}}{\omega(\|f-g\|_{BV(\sigma)})}, \sigma\right) \\ &\leq \frac{\|\mathcal{H}f - \mathcal{H}g\|_{BV(\sigma)}}{\omega(\|f-g\|_{BV(\sigma)})} Var\left(\frac{\mathcal{H}f - \mathcal{H}g}{\|\mathcal{H}f - \mathcal{H}g\|_{BV(\sigma)}}, \sigma\right) \leq 1. \end{aligned}$$

This means that

$$\frac{cVar\left(\frac{\mathcal{H}f - \mathcal{H}g}{\omega(\|f-g\|_{BV(\sigma)})}, \gamma, \sigma\right)}{\ell(\gamma)} \leq 1 \quad \text{for all} \quad \gamma \in \Gamma,$$

Which, in turn, implies that

$$\left|\frac{\mathcal{H}f - \mathcal{H}g}{\omega(\|f-g\|_{BV(\sigma)})}(z) - \frac{\mathcal{H}f - \mathcal{H}g}{\omega(\|f-g\|_{BV(\sigma)})}(\hat{z})\right| \leq \ell(\gamma) \quad \text{for all} \quad \gamma \in \Gamma,\ z, \hat{z} \in \gamma([0,1]),$$

or, equivalently,

$$|(\mathcal{H}f - \mathcal{H}g)(z) - (\mathcal{H}f - \mathcal{H}g)(\hat{z})| \leq \omega(\|f-g\|_{BV(\sigma)})\ell(\gamma) \tag{3.15}$$

for all $\gamma \in \Gamma$, $z, \hat{z} \in \gamma([0, 1])$.

Step 2. Now we will show that the generating function h is also continuous in the second variable. For arbitrarily fixed $z \in \sigma$, by (3.14) and (3.15), we have

$$\begin{aligned} |h(z, y_1) - h(z, y_2)| &= |\mathcal{H}(f_1)(z) - \mathcal{H}(f_2)(z)| \\ &= |(\mathcal{H}(f_1) - \mathcal{H}(f_2))(z)| \\ &\leq \|\mathcal{H}(f_1) - \mathcal{H}(f_2)\|_\infty \\ &\leq \|\mathcal{H}(f_1) - \mathcal{H}(f_2)\|_{BV(\sigma)} \\ &\leq \omega(\|f_1 - f_2\|_{BV(\sigma)}) \\ &= \omega(|y_1 - y_2|). \end{aligned}$$

Step 3. Here we will show that h satisfies the Jensen functional equation in the second variable.

Let $\gamma \in \Gamma$ and let $z1, z2 \in \gamma([0, 1])$ be such that $z1 \leq z2$ (see remark 3.5). Define the function

$$\eta_\gamma(z) = \begin{cases} 0 & \text{if} \quad \gamma(0) \leq z \leq z_1 \\ \dfrac{z - z_1}{z_2 - z_1} & \text{if} \quad z_1 < z \leq z_2 \\ 1 & \text{if} \quad z_2 < z \leq \gamma(1). \end{cases}$$

Now consider $y_1, y_2 \in C$ such that $y_1 \neq y_2$ and define two auxiliary functions as follows:

$$\begin{aligned} f_1(z) &:= \frac{1}{2}\left[\eta_\gamma(z)(y_1 - y_2) + y_1 + y_2\right] \\ f_2(z) &:= \frac{1}{2}\left[\eta_\gamma(z)(y_1 - y_2) + 2y_2\right]. \end{aligned}$$

Notice that

$$f_1(z) - f_2(z) = \frac{y_1 - y_2}{2}$$

and hence $Var(f_1 - f_2, \sigma) = 0$.

On the other hand

$$\begin{aligned} f_1(z_1) &= \frac{1}{2}\left[\eta_\gamma(z_1)(y_1 - y_2) + y_1 + y_2\right] = \frac{y_1 + y_2}{2}, \\ f_1(z_2) &= \frac{1}{2}\left[\eta_\gamma(z_2)(y_1 - y_2) + y_1 + y_2\right] = y_1, \\ f_2(z_1) &= \frac{1}{2}\left[\eta_\gamma(z_1)(y_1 - y_2) + 2y_2\right] = y_2, \\ f_2(z_2) &= \frac{1}{2}\left[\eta_\gamma(z_2)(y_1 - y_2) + 2y_2\right] = \frac{y_1 + y_2}{2}. \end{aligned}$$

Therefore

$$\mathcal{H}f_1(z_1) = h(z_1, f_1(z_1)) = h\left(z_1, \frac{y_1+y_2}{2}\right) \tag{3.16}$$
$$\mathcal{H}f_1(z_2) = h(z_2, f_1(z_2)) = h(z_2, y_1) \tag{3.17}$$
$$\mathcal{H}f_2(z_1) = h(z_1, f_2(z_1)) = h(z_1, y_2) \tag{3.18}$$
$$\mathcal{H}f_2(z_2) = h(z_2, f_2(z_2)) = h\left(z_2, \frac{y_1+y_2}{2}\right) \tag{3.19}$$

Thus, by (3.15) it follows that if γ is the line segment [z1, z2],

$$|(\mathcal{H}f_1 - \mathcal{H}f_2)(z_1) - (\mathcal{H}f_1 - \mathcal{H}f_2)(z_2)| \le \omega(\|f_1 - f_2\|_{BV(\sigma)})\ell(\gamma)$$
$$|(\mathcal{H}f_1 - \mathcal{H}f_2)(z_1) - (\mathcal{H}f_1 - \mathcal{H}f_2)(z_2)| \le \omega(|y_1 - y_2|)\ell(\gamma).$$

Hence,

$$|\mathcal{H}f_1(z_1) - \mathcal{H}f_2(z_1) - \mathcal{H}f_1(z_2) + \mathcal{H}f_2(z_2)| \le \omega(|y_1 - y_2|)\ell(\gamma)$$

Or

$$|h(z_1, f_1(z_1)) - h(z_1, f_2(z_1)) - h(z_2, f_1(z_2) + h(z_2, f_2(z_2))| \le \omega(|y_1 - y_2|)\ell(\gamma)$$

This, by virtue of identities (3.16) through (3.19) implies.

$$\left|h\left(z_1, \frac{y_1+y_2}{2}\right) - h(z_1, y_2) - h(z_2, y_1) + h\left(z_2, \frac{y_1+y_2}{2}\right)\right| \le \omega(|y_1 - y_2|)\ell(\gamma)$$

Making now $z_1 \to z_2$, the continuity of ω at zero and the continuity of h in the first variable imply that (since $\ell(\gamma) \to 0$ as $z_1 \to z_2$)

$$2h\left(z_2, \frac{y_1+y_2}{2}\right) - h(z_2, y_1) - h(z_2, y_2) = 0.$$

Thus, as claimed, h satisfies the Jensen functional equation in the second variable.

From the continuity of h in the second variable we deduce, by Theorem 3.4, that there exist a additive function A(z2) : C → C and a complex number B(z2) such that

$$h(z_2, w) = A(z_2)w + B(z_2), \qquad w \in C. \tag{3.20}$$

Since (3.20) holds for all z2 ∈ σ([0, 1]) we conclude that

$$h(z, w) = A(z)w + B(z), \qquad w \in C, z \in \sigma. \tag{3.21}$$

Finally, notice that if 0 ∈ C, then, by taking y=0 in (3.21), we must have h(z, 0) = B(z), for all z ∈ σ, which implies that B ∈ BV (σ).

4. CONFLICT OF INTERESTS

The authors declare they have no competing interests.

5. AUTHORS CONTRIBUTION

All the authors have contributed equally, read, and approved the submitted paper.

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7. CONCLUSIONES:

En este artículo damos condiciones sobre el operador de superposición H, definido sobre un espacio de funciones de variación acotada sobre subconjuntos compactos de C bajo las cuales la función generadora h es lineal en la variable funcional.

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