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# DISCRETIZING THE HOPF–HOPF BIFURCATION

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**Abstract.** *In this presentation we study the one-step discretization of ODEs with a Hopf–Hopf Bifurcation. We analyze how the local bifurcation diagram is perturbed by the one-step methods. The numerical approximation of the critical eigenvalues is also discussed. We interpret the obtained results in terms of perturbation of branching points. We illustrate the main results by means of numerical experiments.*

**Key–Words.** bifurcation problems, ordinary differential equations, one-step methods.

**Resumen.** *En este artículo se estudia la discretización de un paso de EDOs con una bifurcación Hopf-Hopf. Nosotros analizamos como el diagrama de bifurcación local es perturbado por los métodos de un paso. Se discute también la aproximación numérica de los valores propios críticos. Los resultados son interpretados en términos de perturbación de puntos de ramificación. Los resultados principales se ilustran a traves de experimentos numéricos.*

**Palabras claves:** Problemas de bifurcación, ecuaciones diferenciales ordinarias, método 1-paso.

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## 1. INTRODUCTION

Consider the one-step method with step-size $h$

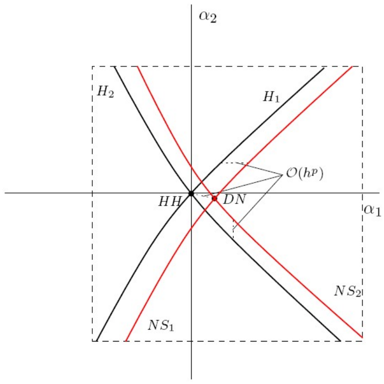
$$x_n = \psi^h\left(x_{n-1}, \alpha\right), \quad n \in \mathbb{N} \qquad \textbf{(1)}$$

which approximates the evolution operator of the parametrized family of continuous-time dynamical systems

$$\dot{x}(t) = f\left(x(t), \alpha\right), \qquad \textbf{(2)}$$

where $f \in C^k\left(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R}^N\right)$ and $k \geq 1$ is sufficiently large. In many practical applications the only way to explore the dynamics of (2) is via numerical methods. For this purpose, we can use the numericaltime integration given by (1). This approach is often referred to as the *indirect method*, cf. [2]. Thus, it is important to investigate whether, and to what extent, the numerical data produced by (1) accurately represents the dynamics of (2). This analysis becomes more involved if the system undergoes bifurcations under variation of parameters.
In this work we assume that (2) has a Hopf–Hopf bifurcation, which occurs when the linearization of (2) about an equilibrium has two pairs of pure imaginary eigenvalues. Furthermore, we suppose that (2) is discretized via general one-step methods of order $p \geq 1$, see Section 2.
The results we want to show are illustrated in Figure 1. In this picture, the curves labeled $H_{1,2}$ and $NS_{1,2}$ represent paths of Hopf points of (2) and Neimark–Sacker points of (1), respectively. The labels *HH* and *DN* stand for a Hopf–Hopf point of (2) and for the resulting double Neimark–Sacker point of the one-step method, respectively.

**FIGURE 1**
***Discretizing the hopf–hopf bifurcation***
**Discretization of the Hopf curves near an HH bifurcation**




## 2. BASIC SETUP

In this presentation we discretize (2) via the one-step map

$$\psi^h(x, \alpha) := x + h\,\Phi\left(h, x, \alpha\right), \ (3)$$

with $\Phi : \left[-h^*, h^*\right] \times \widetilde{\Omega} \times \widetilde{\Lambda} \to \mathbb{R}^N$ sufficiently smooth, $h^* > 0$, where $\widetilde{\Omega} \subset \mathbb{R}^N$ and $\widetilde{\Lambda} \subset \mathbb{R}^2$ are compact sets. We say that (3) is of order $p \geq 1$ if there exists a positive constant $K$ (depending only on f) such that

$$\left\| \phi^h(x,\alpha) - \psi^h(x,\alpha) \right\| \leq K\left|h\right|^{|p+1|}$$

holds for all $\left(h, x, \alpha\right) \in \left[-h^*, h^*\right] \times \widetilde{\Omega} \times \widetilde{\Lambda}$, where $\phi^h\left(\cdot, \alpha\right)$ represents the t-flow of (2). It can be shown

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that there exist smooth functions $\Upsilon, \Xi : [-h_0, h_0] \times \widetilde{\Omega} \times \widetilde{\Lambda} \to \mathbb{R}^N$ such that:

$$\begin{aligned} \psi^h(x,\alpha) &= \phi^h(x,\alpha) + \Upsilon(h,x,\alpha)\, h^{p+1} \quad \text{and} \\ \Phi(h,x,\alpha) &= f(x,\alpha) + \Xi(h,x,\alpha)\, h \end{aligned} \tag{4}$$

hold for all $(h,x,\alpha) \in [-h_0, h_0] \times \widetilde{\Omega} \times \widetilde{\Lambda}$, where $0 < h_0 < h^*$, see [4].

An HH point lies generically at the transversal intersection of two curves of Hopf points. The presence of such a point in (2) may produce a very complex system response. Depending on the values of the normal form coefficients (cf. [3, Section 8.6.2]), the system can have invariant tori and chaotic dynamics, as well as Neimark–Sacker bifurcations of cycles and Shil'nikov homoclinic bifurcations. Our main concern in this presentation is to analyze the effect of onestep methods on an HH point and on the intersecting Hopf curves.

A generic HH bifurcation is a regular zero of the real form of the system (cf. [1])

$$\begin{cases} f(x,\alpha)=0, \\ f_x(x,\alpha)\varphi_1 - iw_1\varphi_1 = 0, \\ \langle l_1,\varphi_1\rangle - i = 0, \\ f_x(x,\alpha)\varphi_2 - iw_2\varphi_2 = 0, \\ \langle l_1,\varphi_1\rangle - i = 0, \end{cases} \tag{5}$$

where $\varphi_{1,2}$ are eigenvectors corresponding to the critical eigenvalues $iw_{1,2}$ and $l_{1,2}$ are suitably chosen normalizing vectors. In this system we use the standard inner product $\langle p,q\rangle := \langle p,q\rangle_{\mathbb{C}^N} = \bar{p}^T q$.

In a similar way, a double Neimark–Sacker point (the "discrete version" of an HH point) of the onestep map (3) can be seen as a solution of the following complex system

$$\begin{cases} \frac{1}{h}\left(\psi^h(x,\alpha) - x\right) = 0, \\ \frac{1}{h}\left(\psi_x^h(x,\alpha)\varphi_1 - e^{ihw_1}\varphi_1\right) = 0, \\ \langle l_1,\varphi_1\rangle - i = 0, \\ \frac{1}{h}\left(\psi_x^h(x,\alpha)\varphi_2 - e^{ihw_2}\varphi_2\right) = 0, \\ \langle l_2,\varphi_2\rangle - i = 0, \end{cases} \tag{6}$$

Define $z := \left(\mathrm{Re}(\varphi_{1,2}), \mathrm{Im}(\varphi_{1,2}), w_{1,2}\right) \in \mathbb{R}^{4N+2}$ and write the systems (5) and (6) as $F(x,\alpha,z) = 0$ and $G(x,\alpha,z) = 0$, respectively. By (4), we can show that

$$G(x,\alpha,z) = F(x,\alpha,z) + O(h)$$

holds for all $(h,x,\alpha,z) \in [-h_0,h_0] \times \widetilde{\Omega} \times \widetilde{\Lambda} \times \mathbb{R}^{4N+2}$. This relation, combined with the implicit function theorem, allows us to prove that an HH point is turned into a DN point by the one-step methods.

Furthermore, the DN point of the one-step map varies smoothly with the step-size and remains close to the original HH point, provided the step-size is sufficiently small. This fact can also be discussed in the context of perturbation of simple branching points. According to [3, Lemma 10.3], a generic HH bifurcation of (2) is a simple branching point (in short SB point) of the system

$$Q(x,\alpha) := \begin{pmatrix} f(x,\alpha) \\ \det\left(2 f_x(x,\alpha) \odot I_N\right) \end{pmatrix} = 0 \tag{7}$$

Here, the symbol $\odot$ stands for the bialternate product of matrices (cf. [3, Section 10.7]). On the other hand, a DN bifurcation of (3) is an SB point of the system

$$P(h,x,\alpha) := \begin{pmatrix} \frac{1}{h}\left(\psi^h(x,\alpha) - x\right) \\ \det\left(\frac{1}{h}\left(\psi_x^h(x,\alpha) \odot \psi_x^h(x,\alpha) - I_m\right)\right) \end{pmatrix} = 0$$

where $m := \frac{1}{2} N(N-1)$. The functions $Q$ and $P$ defined above satisfy locally the relation

$$P(h,x,\alpha) = Q(x,\alpha) + O(h).$$

This means that the SB point of (7) is stable under smooth $O(h)$-perturbations produced by the one-step methods. It is important to point out that we have not assumed any special structure of the vector field (2), so the fact described above is not a consequence of the preservation of any symmetry but of the transversal intersection of the emanating Hopf curves.

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