matemática: Una publicación del ICM – ESPOL
2011, Vol. 9, No. 2

STARTING HOMOCLINIC TANGENCIES NEAR 1:1 RESONANCES

Páez Joseph[1]

Abstract. *In this presentation we construct a theory-based numerical method for starting the continuation of homoclinic tangencias near 1 : 1 resonances, for systems with arbitrary dimension ≥ 2. The method is based on numerical center manifold reduction and flow approximation. The effectiveness of the method is illustrated by numerical examples.*

Keywords: Homological equation, homoclinic orbits, numerical continuation.

Resumen. *En esta presentación se construye un método numérico basado en la teoría, para la inicialización de la continuación de tangencias homoclinas cerca de resonancias 1:1, para sistemas de dimension arbitraria ≥ 2. El método se basa en la reducción numérica a la variedad central y la aproximación a través de flujos. La efectividad del método se ilustra con un ejemplo numérico.*

Palabras claves: ecuación nomológica, órbitas homoclinas, continuación numérica.

RECIBIDO: Agosto 2011
ACEPTADO: Septiembre 2011

1. INTRODUCTION

A typical problem in the numerical analysis of homoclinic orbits is the choice of an appropriate initial guess that could lead us, via e.g. Newton iterations, to the homoclinic connection we want to analyze. In our case we have a well-posed problemgiven in terms of an operator, whose solutions correspond to the numerical approximation of the homoclinic tangencies we are interested in (see e.g. [1]). Thus we will construct a theory-based starting procedure, by means of which we can obtain an "educated" initial guess of the solution of the underlying well-posed problem. What is commonly done is to set a first approximating orbit to

$$\left(...,\xi,...,\xi,x_1,...,x_r,\xi,...,\xi,...\right),$$

where ξ represents an equilibriumpoint and $x_i, i=1,...,r, r\in\mathbb{N}$ are, basically, randomly chosen vectors on the state space. This method is successfully applied, e.g., in [1]. This is of course a purely trial-and-error-based method, where the user relies entirely on "luck" or brute force. Therefore, the user has essentially no control over the outcome of the Newton iterations, and it can easily happens that a spurious solution is obtained, which is a significant disadvantage. Thus, our main concern throughout this presentation will be the construction of a theory-based method that allows us to start the continuation of tangential homoclinic orbits near 1 : 1 resonances, with no restriction of the dimension of the system. This method will enable us to quantitatively explore the homoclinic structure in various interesting examples, since homoclinic orbits are one of the most fascinating objects of study in the theory of dynamical systems, because their presence leads to nontrivial dynamics.

2. BASIC SETUP

Let $f\left(\cdot,\alpha\right), f\in C^k\left(\mathbb{R}^N\times\mathbb{R}^2,\mathbb{R}^N\right), k\geq 1,$ be a diffeomorphism for all $\alpha\in\mathbb{R}^2$. Throughout this presentation we consider the discrete-time system

$$x\mapsto f\left(x,\alpha\right) x\ 7\to f(x,\text{a}). \quad (1)$$

Suppose that $\xi\in\mathbb{R}^N$ is a hyperbolic equilibriumof (1) at $\alpha=\alpha_0$. An orbit $x_{\mathbb{Z}}\in\left(\mathbb{R}^N\right)^{\mathbb{Z}}$ of (1) is referred to as homoclinic with respect to ξ (in short homoclinic) if

$$\lim_{n\to\pm\infty} x_n=\xi$$

Further, a homoclinic orbit $x_{\mathbb{Z}}$ (1) is referred to as tangential, if the stable and unstable manifolds of ξ intersect tangentially along the connecting orbit $x_{\mathbb{Z}}$. We will restrict our study to tangential homoclinic orbits near 1: 1 resonances. We say that the point $\left(x_{\mathbb{Z}},\alpha_{\mathbb{Z}}\right)\in\mathbb{R}^N\times\mathbb{R}^2$ is a 1 : 1 resonance of (1), if $f\left(x_{\mathbb{R}},\alpha_{\mathbb{R}}\right)=x_{\mathbb{R}}$ and the matrix $f_x\left(x_{\mathbb{R}},\alpha_{\mathbb{R}}\right)$ has a double, defective eigenvalue $\mu_{1,2}=1$ and no other eigenvalues on the unit circle.
The normal formof the 1 : 1 resonance is known to be (cf. [2, Lemma 9.7])

[1]*Joseph Páez Chávez, Ph.D, Profesor del Instituto de Ciencias Matemáticas, ESPOL. (e_mail: jpaez@espol.edu.ec)*

$$\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}\mapsto N_\nu(\xi):=\begin{pmatrix}\xi_1+\xi_2\\ \xi_2+\nu_1+\nu_2\xi_2+A(\nu)\xi_1^2+B(\nu)\xi_1\xi_2\end{pmatrix}+O\left(\|\xi\|^3\right),$$

where $A(\cdot), B(\cdot)$ depend smoothly on $\nu:=(\nu_1,\nu_2)\in\mathbb{R}^2$, and $A(0), B(0)\neq 0, \xi_{1,2}\in\mathbb{R}$. This normal form can be approximated by the flow of a vector field for all sufficiently small $\|\nu\|$, that is

$$N_\nu(\xi)=\varphi_\nu^1(\xi)+O\left(\|\nu\|^2\right)+O\left(\|\xi\|^2\|\nu\|\right)+O\left(\|\xi\|^3\right),$$

where φ_ν^t is the flow of a smooth planar system, whose dynamics are described by the Bogdanov-Takens bifurcation theory. Therefore, a tangential homoclinic orbit of the normal form can be approximated by a homoclinic orbit of the vector field, for which several starting procedures are available, see e.g. [3]. Once we have constructed an approximating homoclinic orbit for the normal form of the 1 : 1 resonance, we can transform this orbit to an approximating orbit for a general system (1), via numerical center manifold reduction. For this purpose, the homological equation (cf. [2]) plays a central role. This equation has the form

$$f\left(H(\xi,\nu),K(\nu)\right)=H\left(N_\nu(\xi),\nu\right), \quad \textbf{(2)}$$

where $H:\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}^N$ and $K:\mathbb{R}^2\to\mathbb{R}^2$ are locally defined, smooth functions that represent a parametrization of a center manifold of (1) and a parameter transformation, respectively. By (2), we can obtain linear approximations of the functions *H*,*K*, and hence a homoclinic orbit of the normal form can be transformed into an approximating homoclinic orbit of the general system (1). The resulting starting procedure can be found in [4].

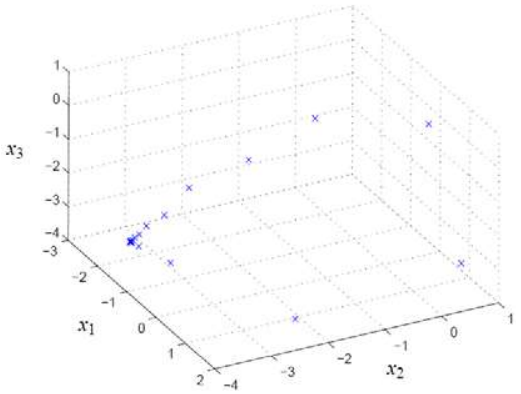
3. NUMERICAL EXAMPLES

Consider the following three-dimensional version of the Hénon map

$$\begin{pmatrix}x\\ y\\ z\end{pmatrix}\mapsto\begin{pmatrix}\alpha_2+\alpha_{1z}-x^2\\ x\\ y\end{pmatrix},$$

This system undergoes a 1: 1 resonance at (*x*,*y*, *z*) = (−0.75,−0.75,−0.75), (α_1,α_2) = (−0.5,−0.5625). With the procedure described in the previous section, we find a homoclinic tangency at (α_1,α_2) ≈(−1.144,−0.233). In the following picture we show the phase plot of the computed orbit.

FIGURE 1
Starting homoclinic tangencies near 1:1 resonances
Tangential homoclinic orbit of the h´enon map



4. ACKNOWLEDGMENT

The author is deeply indebted to Wolf-Jürgen Beyn and Thorsten Hüls for stimulating discussions and suggestions about the present work.

REFERENCES AND ELECTRONIC

[1]. Kleinkauf, J.-M., 1997. "Numerische Analyse tangentialer homokliner Orbits". PhD thesis, University of Bielefeld, Germany.

[2]. Kuznetsov, Y. A., 2004. *Elements of Applied Bifurcation Theory*, third ed., Vol. 112 of *Applied Mathematical Sciences*. Springer-Verlag, New York.

[3]. Beyn,W.-J., 1994. "Numerical analysis of homoclinic orbits emanating from a Takens-Bogdanov point". *IMA Journal of Numerical Analysis,* ***14***, pp. 381–410.

[4]. Páez Chávez, J. Starting Homoclinic Tangencies near 1:1 Resonances. To appear in *Internat. J. of Bif. and Chaos*.