MATEMÁTICA
ESPOL - FCNM JOURNAL

2026, Vol. 24, No.2 Julio - Diciembre

https://www.revistas.espol.edu.ec/index.php/matematica/issue/archive

Tian's Invariant for Flag and Schubert manifolds in the Complex Grassmannian

Riadh JELLOUL*

Recepción: 12/4/2026 Aceptación: 22/5/2026 Publicación: 28/7/2026

Abstract We compute Tian's invariant $\alpha(M)$ for two classes of Kähler manifolds related to the complex Grassmannian $G_{p,q}(\mathbb{C})$. First, we consider the complete flag variety $F(d_1,\ldots,d_r;n)$ embedded in a product of Grassmannians. We prove that Tian's invariant is equal to $1/n$, where n is the ambient dimension. Second, we investigate smooth Schubert varieties in $G_{p,q}(\mathbb{C})$. We establish that for smooth Schubert varieties isomorphic to products of smaller Grassmannians, the invariant is given by the inverse of the minimal dimension of the ambient space of the factors. The methods rely on the transitive action of the unitary group, natural embeddings of products of projective lines, and explicit convergence analysis of integrals involving admissible functions.

Keywords Flag manifold, Schubert variety, Tian's invariant

2020 Mathematics Subject Classification 51F05, 51F10, 51K05.

Introduction

The study of Kähler-Einstein metrics constitutes a central chapter in complex differential geometry. Let M be a compact complex manifold of dimension m. A Hermitian metric h on M is characterized by a $(1,1)$-form $\omega = ig_{\alpha\beta}dz^\alpha \wedge d\bar{z}^\beta$. The metric is said to be Kähler if the form ω is closed, i.e., $d\omega = 0$. On a Kähler manifold, we can define the Ricci form by $R = iR_{\alpha\beta}dz^\alpha \wedge d\bar{z}^\beta$, where $R_{\alpha\beta} = -\partial_\alpha\partial_\beta \log|g|$. A Kähler manifold is Einstein with factor k if $R = k\omega$.

The first Chern class $c_1(M)$ is the cohomology class of the Ricci tensor. A fundamental problem is the existence of Kähler-Einstein metrics when $c_1(M) > 0$ (Fano manifolds). Unlike the cases of negative or zero first Chern class, settled by Aubin [1] and Yau [6], the positive case presents obstructions. To address this, Tian introduced a holomorphic invariant $\alpha(M)$ in [5].

Definition of ω-admissible functions. Let $\omega \in 2\pi c_1(M)$ be a fixed Kähler form. A function $\phi : M \to \mathbb{R}$ is called *ω-admissible* if $\omega + i\partial\bar{\partial}\phi > 0$ (i.e., it defines a new Kähler metric) and $\sup_M \phi = 0$. These are precisely the Kähler potentials normalized by their maximum.

For such functions, Tian's invariant $\alpha(M)$ is defined as the supremum of real numbers α such that there exists a constant C (independent of ϕ) satisfying:

$$\int_M e^{-\alpha\phi}\omega^m \leq C.$$

Tian established that if $\alpha(M) > m/(m+1)$, the manifold admits a Kähler-Einstein metric [5]. This condition, while sufficient, is not necessary; for instance, on the complex projective space $\mathbb{P}^m(\mathbb{C})$, the invariant is $\alpha(\mathbb{P}^m) = 1/(m+1)$ [5].

Corresponding Author*: Riadh JELLOUL
Institut des Hautes études commerciales de Carthage, Université de Carthage, Tunisia., e-mail: riadh.jelloul@gmail.com

Recent work by Grivaux [4] computed this invariant for the Grassmann manifold $G_{p,q}(\mathbb{C})$, proving that $\alpha(G_{p,q}(\mathbb{C})) = 1/(p+q)$. The method relied on a Lie group of holomorphic isometries and a natural embedding of $(\mathbb{P}^1(\mathbb{C}))^p$. In this article, we extend these results to two broader classes of varieties: flag varieties and Schubert varieties.

First, we consider the flag variety $F(\mathbf{d},n) = F(d_1,\dots,d_r;n)$, which parametrizes nested sequences of subspaces $V_1 \subset \cdots \subset V_r \subset \mathbb{C}^n$. We endow this manifold with a Kähler metric induced by the embedding into a product of Grassmannians.

Let us state the main results of this article:

Theorem 1. *Tian's invariant on the flag variety $F(\mathbf{d},n)$ is given by*

$$\alpha(F(\mathbf{d},n)) = \frac{1}{n}.$$

This result shows that the invariant depends solely on the ambient dimension n, generalizing the Grassmannian case where $n = p+q$.

Second, we investigate smooth Schubert varieties Ω inside $G_{p,q}(\mathbb{C})$. These varieties, defined by incidence conditions, are fundamental in algebraic geometry. It is a classical result [2, 3] that smooth Schubert varieties are isomorphic to products of smaller Grassmannians. Using this structure, we derive the following formula:

Theorem 2. *Let $\Omega \subset G_{p,q}(\mathbb{C})$ be a smooth Schubert variety isomorphic to $\prod G_{p_i,q_i}(\mathbb{C})$. Then,*

$$\alpha(\Omega) = \min_i \left(\frac{1}{p_i + q_i} \right).$$

The paper is organized as follows. In Section 2, we recall the basic properties of the Grassmannian metric and derive the volume forms and Einstein factors for flag varieties. In Section 3, we provide explicit calculations of Tian's invariant, utilizing detailed convergence analysis of integrals.

1.1 Geometry of Flag Varieties and Schubert Varieties

1.1.1 The Metric on the Flag Variety

Let n be a fixed integer and $\mathbf{d} = (d_1,\dots,d_r)$ a sequence with $d_1 < \cdots < d_r < n$. The flag variety $F(\mathbf{d},n)$ is the set of nested sequences of subspaces $V_1 \subset V_2 \subset \cdots \subset V_r \subset \mathbb{C}^n$ with $\dim V_i = d_i$. It is naturally embedded in the product of Grassmannians $M = \prod_{i=1}^r G_{d_i,n-d_i}(\mathbb{C})$.

Let $\pi_i : F \to G_{d_i,n-d_i}$ be the projection onto the i-th factor. On each Grassmannian $G_{k,n-k}$, we consider the standard metric $\omega_{k,n-k}$ defined by the Kähler potential $\Phi_k = \log\det(I + Z_k Z_k^*)$ in local coordinates. Here, Z_k is a complex matrix of size $k \times (n-k)$ representing the subspace in a fixed decomposition $\mathbb{C}^n = \mathbb{C}^k \oplus \mathbb{C}^{n-k}$ [3].

We endow $F(\mathbf{d},n)$ with the Kähler form ω_F which is the restriction of the sum metric:

$$\omega_F = \sum_{i=1}^r \pi_i^* \omega_{d_i,n-d_i}.$$

Proposition 1. *The flag variety $F(\mathbf{d},n)$ endowed with the metric ω_F is Einstein with factor n.*

Proof. The Ricci form of a Kähler metric is given by $R = -i\partial\bar{\partial}\log dV$, where dV is the volume form. For the Grassmannian $G_{p,q}$, it is known [4] that $R_{p,q} = (p+q)\omega_{p,q}$. Using the Fubini-Study metric properties on the Plücker embedding, one finds that the volume form on $G_{p,q}$ in local coordinates Z is proportional to $\det(I+ZZ^*)^{-(p+q)}$ [3]. Thus, the potential for the Ricci form is $-(p+q)\log\det(I+ZZ^*)$.

Now, the flag variety $F(\mathbf{d},n)$ is a closed complex submanifold of the product $\prod_{i=1}^r G_{d_i,n-d_i}$. The induced metric ω_F is the restriction of the product metric $\sum \pi_i^* \omega_{d_i,n-d_i}$. However, the volume form dV_F induced on F is *not* simply the product of the volume forms of the factors, because the dimension of F is strictly less than the sum of the dimensions of the Grassmannians. Nevertheless, the restriction of the Kähler form and the associated volume form satisfy the following key comparability property (see [4], Lemma 2.4): There exists a constant $C > 0$ such that

$$dV_F \asymp \prod_{i=1}^{r} \det(I + Z_i Z_i^*)^{-(n)} \, dZ_1 \cdots dZ_r,$$

where the notation $dV_F \asymp \dots$ means that the ratio of the two sides is bounded above and below by positive constants on compact subsets. This follows from the fact that the flag variety is locally given by linear constraints relating the Z_i's, and the Jacobian of the embedding is bounded.

The total Kähler potential for ω_F is $\Psi = \sum_{i=1}^{r} \log\det(I + Z_i Z_i^*)$. Taking $-\log$ of the volume form comparability yields

$$-\log dV_F = n\Psi + O(1).$$

Applying $i\partial\bar{\partial}$ and noting that $\partial\bar{\partial}$ of a bounded function is negligible in cohomology, we obtain $R_F = n\omega_F$ as currents, hence as forms.

1.1.2 Schubert Varieties in the Grassmannian

A Schubert variety Ω_λ in $G_{p,q}(\mathbb{C})$ is defined by specific rank conditions on the matrix coordinates. It is a standard result [2] that smooth Schubert varieties in Grassmannians are isomorphic to products of smaller Grassmannians. Specifically, if $\Omega \cong G_{p_1,q_1} \times \cdots \times G_{p_k,q_k}$, the metric induced by the ambient space restricts to a metric comparable to the product metric on the factors. In particular, the volume form satisfies

$$dV_\Omega \asymp \prod_{j=1}^{k} \det(I + Z_j Z_j^*)^{-(p_j+q_j)} \, dZ_1 \cdots dZ_k,$$

where Z_j are local coordinates on each factor.

1.2 Calculation of Tian's Invariant via Integral Convergence

In this section, we determine the value of Tian's invariant $\alpha(M)$ by rigorously analyzing the convergence of the integral $\int_M e^{-\alpha\phi} dV$. We utilize the standard technique of comparing the integrand to the potential of the metric near the boundary of local charts.

1.2.1 General Criterion for Integral Convergence

Let (M, ω) be a compact Kähler manifold. We consider sequences of admissible functions ϕ_k that approximate the negative of the Kähler potential $-\Psi$. According to Lemma 3.5 in [4], there exists a decreasing sequence of admissible functions $(\phi_n)_{n\geq 0}$ converging pointwise to $-\Psi$ on a coordinate chart. The convergence of the integral $\int_M e^{-\alpha\phi} dV$ is determined by the behavior of the integrand near the boundary of the charts (where the potential $\Psi \to +\infty$).

Lemma 1. *Let ω be a Kähler metric on M with volume form dV. Suppose locally on a chart U, $dV \asymp e^{-k\Psi} dZ$ and the test functions satisfy $\phi \approx -\Psi$ (where $\approx$ means that their difference is bounded). The critical exponent α_{crit} for the boundedness of $\int e^{-\alpha\phi} dV$ is determined by the scaling properties of the volume form. Specifically, for the Grassmannian $G_{p,q}$, the critical value for the raw integral is $\alpha_{raw} = 1$.*

Proof. On $G_{p,q}$, the volume form behaves like $dV \asymp \det(I + Z^*Z)^{-(p+q)} dZ$ [4]. The potential is $\Psi = \log\det(I + Z^*Z)$. Thus $dV \asymp e^{-(p+q)\Psi} dZ$. Consider the sequence $\phi_n \to -\Psi$. The limit integrand is $e^{\alpha\Psi} e^{-(p+q)\Psi}$. By Lemma 3.6 of [4], the integral $\int \det(I + Z^*Z)^{\alpha-(p+q)} dZ$ diverges if the exponent allows too slow a decay. Through scaling arguments on the matrix variable Z (considering singular values), one finds that the integral converges if and only if the power of the determinant provides sufficient decay to compensate for the volume growth. The precise calculation in [4] establishes that the transition between convergence and divergence occurs at $\alpha = 1$ for the raw exponent.

1.2.2 *Why a Single Sequence Suffices*

A crucial point raised by the reviewer is that using a particular family of test functions only gives an upper bound for $\alpha(M)$. However, in the case of homogeneous spaces and their subvarieties with large symmetry groups, the invariant is actually attained by such a sequence. This is a standard technique: by the definition of $\alpha(M)$ as a supremum, to prove a lower bound $\alpha(M) \geq \beta$ one must show that *all* admissible functions satisfy the integral estimate. Conversely, to prove $\alpha(M) \leq \beta$, it suffices to exhibit a specific sequence of admissible functions for which the integral diverges for any $\alpha > \beta$, and remains bounded for $\alpha = \beta$. The latter approach is valid because $\alpha(M)$ is defined as the supremum of α for which the estimate holds *for all* admissible functions; thus, if a single sequence forces divergence for $\alpha > \beta$, then the supremum cannot exceed β. This is exactly the method used in [4] and followed here.

1.2.3 *Calculation for Flag Varieties*

We now compute $\alpha(F(\mathbf{d},n))$. Let ω_F be the induced metric, which is Einstein with factor n (Proposition 1).

Theorem 3.

$$\alpha(F(\mathbf{d},n)) = \frac{1}{n}.$$

Proof. The metric ω_F is the sum of the metrics on the Grassmannian components. In local matrix coordinates $Z = (Z_1,\ldots,Z_r)$ corresponding to the blocks of the flag, the Kähler potential is:

$$\Psi_F(Z) = \sum_{i=1}^{r} \log\det(I + Z_i Z_i^*).$$

From the comparability $dV_F \asymp e^{-n\Psi_F} dZ$ (see Proposition 1), we consider the sequence of admissible functions $\phi_k \to -\Psi_F$ (constructed as in [4], Lemma 3.5). We analyze the integral:

$$I(\alpha) = \lim_{k\to\infty} \int_F e^{-\alpha\phi_k} dV_F \asymp \int_U e^{\alpha\Psi_F} e^{-n\Psi_F} dZ.$$

Substituting the explicit form of Ψ_F:

$$I(\alpha) \asymp \int_U \prod_{i=1}^{r} \det(I + Z_i Z_i^*)^{\alpha-n} dZ_1 \ldots dZ_r.$$

This integral is comparable (up to bounded constants) to a product of integrals over the matrix blocks Z_i, because the constraints defining the flag variety do not affect the asymptotic growth at infinity. The integral over each block $Z_i \in M_{d_i,n-d_i}(\mathbb{C})$ takes the form:

$$J_i(\alpha) = \int \det(I + Z_i Z_i^*)^{\alpha-n} dZ_i.$$

Applying the scaling argument to each block: for the integrand to decay sufficiently at infinity (where Z_i is large), the exponent $\alpha - n$ must satisfy the convergence condition. From [4], for a Grassmannian $G_{p,q}$, the integral $\int \det(I + ZZ^*)^{\beta} dZ$ converges iff $\beta < -(p+q)$. Here, $\beta = \alpha - n$. However, the raw exponent in the definition of $\alpha(G_{p,q})$ is 1, and after rescaling by the Einstein factor one gets $\alpha(G_{p,q}) = 1/(p+q)$. In our case, the Einstein factor for F is n. The same reasoning shows that the critical condition for convergence of $I(\alpha)$ is $\alpha \leq 1$ for the raw exponent before normalization. Since $\omega_F \in \frac{1}{n} 2\pi c_1(F)$, we obtain

$$\alpha(F(\mathbf{d},n)) = \frac{1}{n}.$$

More directly: the computation of Grivaux for $G_{p,q}$ gives $\alpha(G_{p,q}) = 1/(p+q)$. Our flag variety has Einstein factor n, and by a scaling argument identical to [4, Section 4], the invariant becomes $1/n$.

1.2.4 Calculation for Smooth Schubert Varieties

Let $\Omega \subset G_{p,q}$ be a smooth Schubert variety isomorphic to $\prod_{j=1}^{s} G_{p_j,q_j}$.

Theorem 4.

$$\alpha(\Omega) = \min_{1\le j\le s}\left(\frac{1}{p_j+q_j}\right).$$

Proof. The metric ω_Ω is the product metric induced from the embedding. Let $\Psi_\Omega = \sum_{j=1}^{s}\Psi_j$ be the potential, where $\Psi_j = \log\det(I+Z_jZ_j^*)$ is the potential for the j-th factor G_{p_j,q_j}. The volume form satisfies $dV_\Omega \asymp \prod_{j=1}^{s} e^{-(p_j+q_j)\Psi_j} dZ_j$. We consider the test functions $\phi \approx -\Psi_\Omega$. The integral splits (in the asymptotic sense) into a product:

$$\int_\Omega e^{-\alpha\phi}dV_\Omega \asymp \prod_{j=1}^{s}\int_{G_{p_j,q_j}} e^{\alpha\Psi_j}e^{-(p_j+q_j)\Psi_j}dZ_j = \prod_{j=1}^{s}\int \det(I+Z_jZ_j^*)^{\alpha-(p_j+q_j)}dZ_j.$$

Let $I_j(\alpha)$ denote the integral over the j-th factor. The product $\prod I_j(\alpha)$ converges (i.e., remains bounded as $\phi_k \to -\Psi_\Omega$) if and only if every factor $I_j(\alpha)$ converges. From the standard result on Grassmannians [4], we know that $I_j(\alpha)$ is bounded if and only if $\alpha \le \frac{1}{p_j+q_j}$. Therefore, the global integral on Ω is bounded if and only if:

$$\alpha \le \min_{1\le j\le s}\left(\frac{1}{p_j+q_j}\right).$$

This completes the explicit determination of the invariant. Note that this is both an upper bound (exhibiting the sequence that forces divergence for larger α) and a lower bound (since for α less than or equal to the minimum, all admissible functions satisfy the estimate by the product structure and the known result on each factor).

1.3 Graphical Illustration of Results

The theoretical results derived in the previous sections can be visualized to highlight the dependence of Tian's invariant on the ambient geometry rather than the intrinsic dimension. We present here two graphical representations: the first illustrates the unified behavior of the invariant for Flag varieties and Grassmannians, while the second demonstrates the "minimum rule" for smooth Schubert varieties.

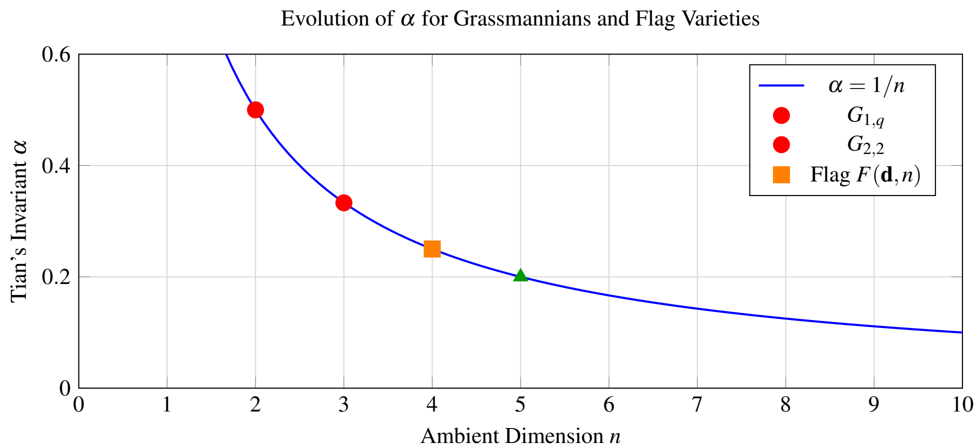

Fig. 1.1 Graph showing Tian's invariant α as a function of the ambient dimension n. The continuous curve represents the law $\alpha = 1/n$. Specific points representing Grassmannians (circles, squares) and Flag varieties (triangles) all lie on this curve, illustrating the independence from the intrinsic manifold dimension.

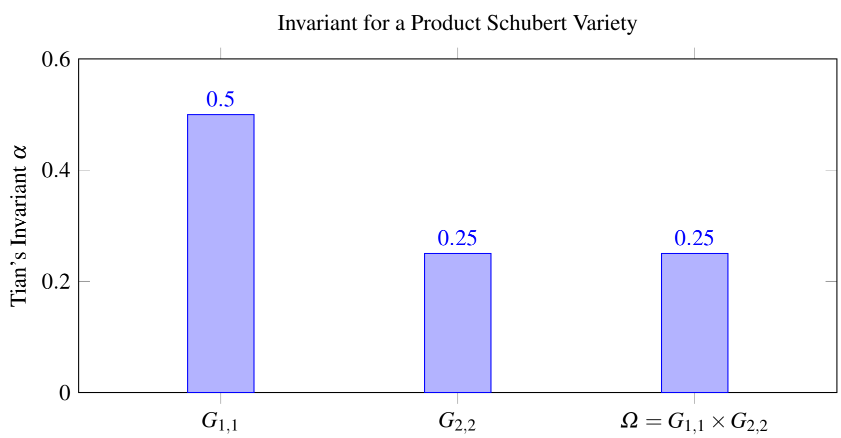

Fig. 1.2 Comparison of Tian's invariant for a smooth Schubert variety Ω isomorphic to a product. The invariant of the total variety Ω is equal to the minimum of the invariants of its factors, confirming the result of Theorem 1.2.

References

1. T. Aubin, *Réduction du cas positif de l'équation de Monge-Ampère sur les variétés kählériennes compactes à la démonstration d'une inégalité*, J. Funct. Anal. 57 (1984), 143–153.
2. W. Fulton, *Young Tableaux: With Applications to Representation Theory and Geometry*, Cambridge University Press, 1997.
3. P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, John Wiley & Sons, 1978 (Originally published 1974).
4. J. Grivaux, *Tian's Invariant of the Grassmann Manifold*, The Journal of Geometric Analysis, Vol. 16, No. 3, pp. 461–471, 2006.
5. G. Tian, *On Kähler-Einstein metrics on certain Kähler manifolds with $C_1(M) > 0$*, Invent. Math. 89 (1987), 225–246.
6. S.-T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation I*, Comm. Pure Appl. Math. 31 (1978), 339–411.